

The Lorentz transformations of the vectors E , B , P , M and the external electric fields from a stationary superconducting wire with a steady current and from a stationary permanent magnet

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In the first part of this paper we review the fundamental difference between the usual transformations of the three-dimensional (3D) vectors of the electric field \mathbf{E} , the magnetic field \mathbf{B} , the polarization \mathbf{P} , the magnetization \mathbf{M} and the Lorentz transformations of the 4D geometric quantities, vectors E , B , P , M , with many additional explanations and several new results. In the second part, we have discussed the existence of the electric field vector E outside a *stationary* superconducting wire with a *steady current* and also different experiments for the detection of such electric fields. Furthermore, a fundamental prediction of the existence of the external electric field vector E from a *stationary* permanent magnet is considered. These electric fields are used for the resolution of the “charge-magnet paradox” with 4D geometric quantities for a qualitative explanation of the Aharonov-Bohm effect in terms of fields and not, as usual, in terms of the vector potential and for a qualitative explanation that the particle interference is not a test of a Lorentz-violating model of electrodynamics according to which a magnetic solenoid generates not only a static magnetic field but also a static electric field.

1. Introduction

1.1 About this paper

Both, in the prerelativistic physics and in Einstein’s formulation of special relativity (SR) [1] the electric and magnetic fields are represented by the 3-vectors $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. The notation is as in [2], i.e. \mathbf{E} and \mathbf{B} are called 3-vectors and they are designated in boldface type. In the whole physical literature after [1] the usual transformations of the 3-vectors \mathbf{E} and \mathbf{B} , the last equations in §6., II. Electrodynamical Part, [1] or, e.g., Eqs. (11.148) and (11.149) in [2], i.e., Eq. (10) here, are always considered to be the relativistically correct Lorentz transformations (LT) (boosts) of \mathbf{E} and \mathbf{B} . Here, in the whole paper, under the name LT we shall only consider boosts. They are first derived by Lorentz [3] and Poincaré [4] (see also two fundamental Poincaré’s papers with notes by Logunov [5]) and independently by Einstein [1] and subsequently derived and quoted in almost every textbook and paper on relativistic electrodynamics. Henceforward, these usual transformations of the 3-vectors

will be called the Lorentz-Poincaré-Einstein transformations (LPET), according to physicists who discovered them. The main feature of the LPET of \mathbf{E} and \mathbf{B} is that *the components of the transformed \mathbf{E}' are expressed by the mixture of components of \mathbf{E} and \mathbf{B} , and similarly for \mathbf{B}'* , Eq. (11.148) in [2]. The electric field \mathbf{E} in one frame is “seen” as slightly changed electric field \mathbf{E}' and an *induced magnetic field* \mathbf{B}' in a relatively moving inertial frame.

However, it has recently been proved both in the tensor formalism and in the geometric algebra formalism [6-11] that *these LPET ARE NOT the LT*. They drastically differ from the LT of the relativistically correct 4D geometric quantities, which represent the electric and magnetic fields. In contrast to the LPET, *the correct LT always transform the 4D algebraic object representing the electric field only to the electric field; there is no mixing with the magnetic field*. This fundamental difference between the LPET of the 3-vectors and the LT of the 4D geometric quantities is considered in the first part of this paper.

It is worth mentioning that some experimentally verifiable consequences of that fundamental difference between the LPET and the LT have been examined in several papers. Thus, it is shown in [7] that the conventional theory with the 3D \mathbf{E} and \mathbf{B} and their LPET (10) yields different values for the motional emf ε for relatively moving inertial observers, $\varepsilon = UBl$ and $\varepsilon = \gamma UBl$, whereas the approach with 4D geometric quantities and their LT (42), i.e., (43), always yields the same value for ε , which is defined as a Lorentz scalar, $\varepsilon = \gamma UBl$. This result is very strong evidence that the usual approach is not relativistically correct. If the experimentalists find a way to measure the emf ε for the considered problem of a conductor moving in a static magnetic field, not only for small velocities, $U \ll c$, they should see that in the laboratory frame $\varepsilon = \gamma UBl$ and not simply $\varepsilon = UBl$. That problem is of a considerable importance in practice. A similar discussion was presented for the Faraday disk in [8]. In [12, 13] the Trouton-Noble paradox is considered. It is shown that in the geometric approach with 4D quantities *the 4D torques* will not appear for the moving capacitor if they do not exist for the stationary capacitor, which means that with 4D geometric quantities the principle of relativity is naturally satisfied and there is not the Trouton-Noble paradox. The same conclusion holds in the low-velocity approximation $\beta \ll 1$, or $\gamma \simeq 1$. Very similar paradox to the Trouton-Noble paradox is Jackson’s paradox. It is discussed in detail in [14]; the second paper is a simpler, more pedagogical, version of the first one. There, in [14], it is also shown that there is no paradox in the approach with 4D geometric quantities and their LT.

The most important experimentally verifiable consequence of the difference between the LPET and the LT refers to the existence of the electric field vector outside a *stationary* superconducting wire with a *steady current* and also outside a *stationary* permanent magnet. The second part of this paper, Secs. 7.1, 7.2, 8, is devoted to that problem. In Sec. 9.2, these electric fields are used for the resolution of the “charge-magnet paradox” in terms of 4D geometric quantities without introducing some “hidden” quantities and without changing the expression for the Lorentz force, but as a 4D geometric quantity, like the expression for the Lorentz force density k_L (67). Furthermore, in Sec. 10, these

electric fields are used for a qualitative explanation of the Aharonov-Bohm effect in terms of fields and not, as usual, in terms of the vector potential and in Sec. 11, for a qualitative explanation that the particle interference is not a test of a Lorentz-violating model of electrodynamics according to which a magnetic solenoid generates not only a static magnetic field but also a static electric field.

The geometric approach to special relativity that is used in, e.g., [6-14], is called the invariant special relativity (ISR). In the ISR, it is considered that in the 4D spacetime the 4D geometric quantities are well-defined both theoretically and *experimentally*; they have an independent physical reality. The principle of relativity is automatically satisfied if the physical laws are expressed in terms of the 4D geometric quantities. It is not so in the SR [1] in which it is considered that the 3D quantities have an independent physical reality. There, the principle of relativity is postulated and it is supposed that it holds for physical laws expressed in terms of the 3D quantities, e.g., Maxwell's equations written in terms of the 3-vectors \mathbf{E} and \mathbf{B} . In the ISR, physical quantities are represented by the abstract, coordinate-free, 4D geometric quantities. In the papers [6-14] these quantities are treated as either tensors as geometric objects, e.g., in [6, 8, 10], or, multivectors in the geometric algebra formalism, e.g., in [7-9], [11-14]. If some basis has been introduced, these coordinate-free quantities are represented as 4D coordinate-based geometric quantities (CBGQs) comprising both components and *a basis*. Every 4D CBGQ is invariant under the passive LT; the components transform by the LT and the basis by the inverse LT leaving the whole CBGQ unchanged. This is the reason for the name ISR. The invariance of a 4D CBGQ under the passive LT reflects the fact that such mathematical, invariant, 4D geometric quantity represents *the same physical quantity* for relatively moving inertial observers, see, e.g., Eqs. (49), (73), (82) and (87) here. Hence, it can be stated that in the ISR only quantities that do not change upon the passive LT have an independent physical reality, both theoretically and *experimentally*. In contrast to it the SR [1] deals with the Lorentz contraction, the time dilation and the LPET of the 3D vectors \mathbf{E} and \mathbf{B} . However, e.g., the rest length and the Lorentz contracted length are not the same 4D quantity for relatively moving observers, since the transformed length $L_0(1 - \beta^2)^{1/2}$ is different than the rest length L_0 , see, e.g., Eq. (89) in Appendix. Rohrlich [15] named the Lorentz contraction and other transformations which do not refer to the same 4D quantity as the “apparent” transformations (AT), whereas the transformations which refer to the same 4D quantity as the “true” transformations, e.g., the LT. Hence, the other name for the ISR is the “True transformations relativity” (“TT relativity”), which is used, e.g., in [16, 17]. As proved in [6-11] and exposed here in Secs. 3 - 3.2 and Sec. 6, *the LPET are also the AT and not the LT*. In the 4D spacetime, as shown in detail in [16, 17], instead of the Lorentz contraction and the time dilation one has to consider the 4D geometric quantities, the distance vector l_{AB} , Eq. (85) here, and the spacetime length, Eq. (86) here, which properly transform under the LT, see Appendix here.

1.2 An outline of this paper

An outline of the present paper is as follows. In Sec. 2, a short review of the geometric algebra formalism is presented. For more detail see [18]. An important result from [19] is mentioned in that section. Namely, *what is essential for the number of components of a vector field is the dimension of its domain. Hence, the usual time-dependent $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$ cannot be the 3-vectors, since they are defined on the spacetime.* They are correctly defined geometric quantities, e.g., vectors (4-vectors in the usual notation) $E(x)$, $B(x)$, where x is the position vector.

Then, in Secs. 3 - 3.2, we discuss the traditional derivation of the LPET of the 3-vectors \mathbf{E} and \mathbf{B} , Eq. (10), and \mathbf{P} and \mathbf{M} , Eq. (16), or Eq. (17). As already stated, the main feature of the LPET is that, e.g., *the transformed \mathbf{E}' is expressed by the mixture of the 3-vectors \mathbf{E} and \mathbf{B} , and similarly for \mathbf{B}' .* As shown in Sec. 3.1, for the derivation of (10) one first makes the identification of the six independent components of $F^{\alpha\beta}$ with six components of the 3-vectors \mathbf{E} and \mathbf{B} , Eq. (3). Then, it is simply argued that six independent components of $F'^{\alpha\beta}$ are the “Lorentz transformed” components E'_i and B'_i , Eq. (7), i.e., *the LPET of the components of \mathbf{E} and \mathbf{B} are derived assuming that they transform under the LT as the components of $F^{\alpha\beta}$ transform*, Eq. (11.148) in [2]. However, it is shown in that section that the identifications (3) and (7) depend on the chosen synchronization and that they are meaningless for some nonstandard synchronization, e.g., the “radio” synchronization, see Eqs. (4) and (5), which means that the LPET (10) (and (16), or (17)) are not the relativistically correct LT. In Sec. 3.2 the same discussion is presented for the derivation of the LPET of \mathbf{P} and \mathbf{M} , Eq. (16), or Eq. (17).

In Sec. 4, the definitions of vectors E , B , Eqs. (21) and (22), and P , M , Eqs. (25) and (26), in terms of F , v and \mathcal{M} , u , respectively, are examined; v is the velocity vector of the observers who measure E and B fields, while u is the velocity vector of a moving medium. It is visible from (22) that *in mathematically correct definitions the vectors $E(x)$ and $B(x)$ are derived from F AND v , i.e., they are defined with respect to the observer.* Similarly, it is visible from (26) that P and M depend not only on \mathcal{M} but on u as well. Furthermore, the basic Lorentz invariant field equation for vacuum with F , Eq.(20), is written in terms of E and B , Eq. (27), i.e., Eqs. (28) and (29). The generalization of these field equations to the electromagnetic field equations for moving media is presented in [20] and also briefly considered in this section. The generalization of (20) to a moving medium is obtained simply replacing F by $F + \mathcal{M}/\varepsilon_0$, which yields Eqs. (30), the primary equations for the electromagnetism in moving media with bivectors $F(x)$ and $\mathcal{M}(x)$. Then, these equations are written with vectors $E(x)$, $B(x)$, $P(x)$ and $M(x)$, Eqs. (33) and (34). As stated in [20], Eq. (30), i.e., Eqs. (33) and (34), comprise and generalize all usual Maxwell’s equations with 3-vectors for moving media. The equations (33) and (34) contain both the velocity vector u of a moving medium and the velocity vector v of the observers who measure E and B fields. They are first reported in [20] and do not appear in the previous literature.

In Sec. 5, the LT of vectors E and B , as 4D geometric quantities, are examined and compared with Minkowski’s results. Note that Minkowski, Sec.11.6 in

his famous paper [21], was the first who introduced vectors (in the usual notation 4-vectors) of the electric and magnetic fields and correctly defined their LT. It is shown that the LT of vectors E and B are obtained by a mathematically correct procedure in the 4D spacetime. As explained in [11], *Minkowski, in Sec. 11.6 in [21], showed that both factors of the vector E , as the product of one bivector and one vector, has to be transformed by the LT.* That fundamental Minkowski's result is reinvented and generalized in [6-11]. Thus, E from (22), $E = F \cdot v/c$, transforms under the active LT, e.g., Eqs. (39) and (40), in such a manner that both F and the velocity of the observer v are transformed by the LT, Eq. (41). These coordinate-free LT yield how vector E transforms under the active LT, Eq. (42). If these transformations are written in the standard basis then the transformations of the components are obtained, Eq. (43). The most important result is that *under the relativistically correct LT the electric field vector E transforms again to the electric field vector E' ; there is no mixing with the magnetic field B .*

In Sec. 6, the LPET of the components of the 3-vectors \mathbf{E} and \mathbf{B} are retrieved using the geometric algebra formalism, i.e., the 4D geometric quantities. If in the transformation of $E = F \cdot v/c$ only F is transformed by the LT, but not the velocity of the observer v , then the LPET of the electric field vector E are obtained, Eqs. (44) and (45). These coordinate-free LPET are also written in the standard basis, Eq. (46), and it is visible that *the components of the transformed E'_F are expressed by the mixture of components of E and B .* As seen from Eq. (47), the same result is obtained for the magnetic field vector B . The comparison of the relation for the LPET of the components of E (46) with the LPET for the components $E_{x,y,z}$ of the 3-vector \mathbf{E} , which are given, e.g. by Eq. (11.148) in [2], explicitly shows that *they are exactly the same transformations.* But, the LPET of the vector E , (44) and (45), are obtained by a mathematically incorrect procedure (*only F is transformed*), which means that *they are not the relativistically correct LT* and consequently, contrary to the general opinion, *the LPET of the 3-vector \mathbf{E} (and \mathbf{B} , \mathbf{P} , \mathbf{M}) ARE NOT THE LT but the AT.*

In Sec. 7.1, the second-order electric fields outside a stationary conductor with steady current are considered. In the usual approaches, e.g., [22-26], there is a magnetic field 3-vector outside a stationary (superconducting) wire with steady current, but, according to the LPET (10), there are both, the slightly changed magnetic field and an induced *second-order external electric field 3-vector for the same but moving wire with steady current.* Similarly, e.g. [22, 23], it is argued that a neutral stationary current loop has only a magnetic moment 3-vector. According to the LPET for the 3-vectors \mathbf{p} and \mathbf{m} , which are the same as (16), that current loop acquires an electric dipole moment (18) as well, if it is moving with uniform 3-velocity \mathbf{U} ($\beta = \mathbf{U}/c$). However, *in the 4D spacetime, the electric and magnetic fields and the dipole moments are not the 3-vectors but the 4D vectors E , B , p , m , which transform under the LT (42), i.e., (43) and not under the LPET of the 3-vectors (10) and (16), or (17).* The electric field vector E (the same for B , p , m) transforms by the LT again to the electric field vector without mixing with B and therefore if E exists for a moving wire with a steady

current, or a moving current loop, it must exist for the same but *stationary* wire or current loop. The determination of the electric field vector for the stationary current-carrying conductor is investigated in detail in [27] and here it is briefly reviewed with some additional explanations. The expression for the current density vector in the rest frame of the wire is given by Eq. (54), whereas the expression for the external second-order electric field vector of the stationary wire with steady current is given by Eq. (55). Observe that in the second paper in [27], the incorrect quadrupole field of the stationary current loop from the published version is replaced by the dipole field. Therefore, henceforward, if referred to [27] I mean that the corrected version has to be taken into account.

In Sec. 7.2, the experiments for the detection of the second-order electric fields outside a stationary conductor with steady current are discussed. This is a new consideration that is not reported in my previous papers. In the measurements [28, 29] a direct contact with the superconducting coil is used and because of that they cannot either support or disprove the theory presented in [27]. In [30], a non-contact method of measuring is used, but in order to “see” the external second-order electric fields the coil used in their experimental setup would need to be a superconducting coil. Recently, [31], the most promising method is proposed and it deals with cold ions. The theory presented in [31] is essentially the same as in my paper [32]. However, both theories, [32] and [31], explicitly use the Lorentz contraction in the derivation of the expression (55) for the external second-order electric field and as such they are not the relativistically correct theories. In [27] the relativistically correct theory is presented, but it seems that Folman, [31], either was not aware of [27] or more believed in the usual approach with the Lorentz contraction and the LPET (10) than to the mathematically correct 4D geometric approach.

In Sec. 8, an essentially new prediction is presented that *a stationary permanent magnet possesses an intrinsic polarization, which induces the external electric field*. This prediction is supported both in the usual Ampèrian approach in which a permanent magnet is an assembly of small current loops and also using the recent fundamentally new results from [33], i.e., the relations (59) and (60), which show that any fundamental particle has not only the intrinsic magnetic dipole moment (MDM) m but also the intrinsic electric dipole moment (EDM) p . Then, in the same way as the MDMs determine the magnetization M of a stationary permanent magnet the EDMs determine its polarization P , which induces an electric field outside a permanent magnet (moving or *stationary*). We suggest that the experimental setup from [31] could be also used for the measurement of that electric field outside a stationary permanent magnet.

In Secs. 9 - 9.2 the “charge-magnet paradox” from [34] is discussed in detail together with the highlight of it from [35] and different resolutions from [36]. The paradox, that in a static electric field a MDM \mathbf{m} is subject to a torque \mathbf{N} in some frames and not in others is stated to be resolved in [34] replacing the conventional Lorentz force (density), Eq. (5) in [34], by Einstein-Laub law, Eq. (6) in [34], which predicts no torque in all frames. In [34], all quantities \mathbf{E} , \mathbf{B} , \mathbf{P} , \mathbf{M} , \mathbf{F} , \mathbf{N} , etc. are the 3-vectors and their transformations are given by the LPET (10) for \mathbf{E} and \mathbf{B} and (16), or (17) for \mathbf{P} and \mathbf{M} , and the same for EDM

\mathbf{p} and MDM \mathbf{m} , including (18). All other approaches from [36] also deal with the 3D quantities and their AT and often introduce some “hidden” quantities. Moreover, the resolutions from [34-36] *depend on the chosen synchronization* and they are meaningless *if only the Einstein synchronization is replaced by the “radio” synchronization*, as can be concluded from Eqs. (4) and (5) in Sec. 3.1. This means that from the viewpoint of the ISR the resolutions from [34-36] are not relativistically correct resolutions. All these treatments from [34-36] are objected in Sec. 9.1.

The treatment of the interaction between a static electric field and a permanent magnet that is presented in Sec. 9.2 differs in two important respects relative to [34-36]. As explicitly shown in the very similar treatments of the Trouton-Noble paradox [12, 13] and Jackson’s paradox [14] in the approach with 4D geometric quantities there is no paradox and the same is shown for the “charge-magnet paradox.” Every 4D geometric quantity, e.g., the Lorentz force vector, the torque bivector is the same quantity for all relatively moving inertial observers and for all bases chosen in them. However, in contrast to the Trouton-Noble paradox [12, 13] and Jackson’s paradox [14] in which the torque bivector is zero in all frames, in this case the torque bivector N is different from zero even in the common rest frame of the considered charge and magnet from [34]. The reason for that important difference is that according to the 4D geometric approach, i.e., the ISR, a stationary permanent magnet possesses not only the magnetization vector M but the polarization vector P as well. In the formulation with the torque bivector N the relations (71), (72) and (73) show that the principle of relativity is naturally satisfied and that there is no paradox. As can be seen from (78), (79) and (81), (82), the same holds for the formulation with the “space-space” torque N_s and the “time-space” torque N_t . It is visible from (78) and (79) that the torque N_s is always determined by the interaction of the EDM p of the permanent magnet and the electric field E , whereas N_t is determined by the interaction of the MDM m and E .

In Sec. 10, it is suggested that the existence of the electric fields from a stationary permanent magnet yields the possibility for the explanation of the Aharonov-Bohm effect in terms of fields and not, as usual, in terms of the vector potential.

In Sec. 11, the results from [37] are discussed. In [37] it is argued that a static electric field generated by a magnetic solenoid is due to Lorentz-violating electrodynamics and that the particle interference is a test of that Lorentz-violating model of electrodynamics. However, according to Secs. 7.1 and 8 from this paper there is a static electric field generated by a magnetic solenoid, see Eq. (55), that is obtained in a completely Lorentz-covariant approach with the 4D geometric quantities, which means that the particle interference is not a test of a Lorentz-violating model of electrodynamics.

In Sec. 12, a discussion and conclusions are presented.

In Appendix, it is briefly described the essential difference between the 4D geometric approach, i.e., the ISR, with the spacetime length and Einstein’s definition of the Lorentz contraction for a moving rod.

2. A brief summary of geometric algebra

The calculations in this paper are performed in the geometric algebra formalism. Physical quantities are represented by the abstract multivectors, or, if some basis has been introduced, these abstract quantities are represented as 4D CBGQs comprising both components and *a basis*. For simplicity and for easier understanding mainly the standard basis, described below, will be used.

Here, for the reader's convenience, we provide a brief summary of geometric algebra. Usually Clifford vectors are written in lower case (*a*) and general multivectors (Clifford aggregate) in upper case (*A*). The space of multivectors is graded and multivectors containing elements of a single grade, *r*, are termed homogeneous and usually written A_r . The geometric (Clifford) product is written by simply juxtaposing multivectors AB . A basic operation on multivectors is the degree projection $\langle A \rangle_r$ which selects from the multivector *A* its *r*- vector part (0 = scalar, 1 = vector, 2 = bivector, ...). The geometric product of a grade-*r* multivector A_r with a grade-*s* multivector B_s decomposes into $A_r B_s = \langle AB \rangle_{r+s} + \langle AB \rangle_{r+s-2} \dots + \langle AB \rangle_{|r-s|}$. The inner and outer (or exterior) products are the lowest-grade and the highest-grade terms respectively of the above series; $A_r \cdot B_s \equiv \langle AB \rangle_{|r-s|}$ and $A_r \wedge B_s \equiv \langle AB \rangle_{r+s}$. For vectors *a* and *b* we have: $ab = a \cdot b + a \wedge b$, where $a \cdot b \equiv (1/2)(ab + ba)$, $a \wedge b \equiv (1/2)(ab - ba)$.

In this paper the notation will not be the same as in the above mathematical presentation. Some vectors will be denoted in lower case, like *u*, *v* (the velocities), *x* (the position vector), whereas some others in upper case, like vectors of the electric and magnetic fields *E* and *B* respectively, the Lorentz force K_L . Bivectors will be denoted in upper case but without subscript that denotes the grade. Thus, for example, the electromagnetic field *F* is a bivector.

In, e.g., [18], one usually introduces the standard basis. The generators of the spacetime algebra (the Clifford algebra generated by Minkowski spacetime) are taken to be four basis vectors $\{\gamma_\mu\}$, $\mu = 0 \dots 3$, satisfying $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \text{diag}(+ - - -)$. This basis, the standard basis, is a right-handed orthonormal frame of vectors in the Minkowski spacetime M^4 with γ_0 in the forward light cone. The γ_k ($k = 1, 2, 3$) are spacelike vectors. The γ_μ generate by multiplication a complete basis for the spacetime algebra: 1, γ_μ , $\gamma_\mu \wedge \gamma_\nu$, $\gamma_\mu \gamma_5$, γ_5 ($2^4 = 16$ independent elements). γ_5 is the right-handed unit pseudoscalar, $\gamma_5 = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3$. Any multivector can be expressed as a linear combination of these 16 basis elements of the spacetime algebra. For all mathematical details regarding the spacetime algebra reader can consult [18]. It is worth noting that the standard basis $\{\gamma_\mu\}$ corresponds, in fact, to the specific system of coordinates, i.e., to Einstein's system of coordinates. In Einstein's system of coordinates the standard, i.e., Einstein's synchronization [1] of distant clocks and Cartesian space coordinates x^i are used in the chosen inertial frame. However different systems of coordinates are allowed in an inertial frame and they are all equivalent in the description of physical phenomena. For example, in [16] two very different, but physically completely equivalent, systems of coordinates, Einstein's system of coordinates and the system of coordinates with a nonstandard synchronization, the "everyday," i.e., the "radio" ("r"), synchronization,

are exposed and exploited throughout the paper. The “r”, synchronization is also used and explained in [38], [17], [27], [33], see also Sec. 3.1 here. For simplicity and for easier understanding we shall mainly deal with the standard basis, but remembering that the approach with 4D geometric quantities holds for any choice of basis in M^4 . Observe that the usual covariant approach, e.g., from the well-known textbooks [2], [26] exclusively deals with *components* implicitly taken in a specific basis, the standard basis.

Here, it is worth mentioning an important result regarding the usual formulation of electromagnetism (as in [2], [26]), which is presented in [19]. This is also mentioned in [11]. It is explained in [19] that an individual vector has no dimension; the dimension is associated with the vector space and with the manifold where this vector is tangent. Hence, what is essential for the number of components of a vector field is the number of variables on which that vector field depends, i.e., the dimension of its domain. This means that the usual time-dependent $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$ cannot be the 3-vectors, since they are defined on the spacetime. That fact determines that such vector fields, when represented in some basis, have to have four components (some of them can be zero). Therefore, we use the term “vector” for the correctly defined geometric quantity, which is defined on the spacetime. However, an incorrect expression, the 3-vector or the 3D vector, will still remain for the usual $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$ from [2], see Eq. (10).

3. The LPET of the 3-vectors \mathbf{E} and \mathbf{B} , \mathbf{P} and \mathbf{M}

3.1 The 3-vectors \mathbf{E} and \mathbf{B} and their LPET

Firstly, we discuss and object the derivation of the LPET of the 3-vectors \mathbf{E} and \mathbf{B} by the identification of the components of \mathbf{E} and \mathbf{B} with the components (implicitly taken in the standard basis) of the electromagnetic field tensor $F^{\alpha\beta}$, as, e.g., in the usual covariant approach [2]. Einstein’s derivation [1] of the LPET of the components of \mathbf{E} and \mathbf{B} is discussed and objected in Sec. 5.3 in [16]. (In [16] the LPET are called the “apparent” transformations.) In the usual covariant approach, e.g., [2], the field-strength tensor $F^{\alpha\beta}$ (only components in the standard basis and not the whole tensor as a 4D geometric quantity) is introduced and defined in terms of the vector potential A^μ , Eq. (11.136) in [2].

However, as already stated, the components are coordinate quantities and they do not contain the whole information about the physical quantity. They do not completely represent a physical quantity that is defined on the 4D spacetime, since a basis of the spacetime is not included. Furthermore, in such formulation the gauge dependent vector potential A^μ (not measurable quantity) is considered to be the primary quantity from which the field-strength tensor $F^{\alpha\beta}$ (a measurable quantity) is derived! In contrast to such usual approach, e.g., [2], it is shown in [12] that in the 4D spacetime the electromagnetic field, the bivector $F = F(x)$, can be taken as the primary quantity for the whole electromagnetism and there is no need for the electromagnetic potentials.

Then, in the usual approaches, the covariant form of the Maxwell equations is written with $F^{\alpha\beta}$ and its dual $*F^{\alpha\beta}$

$$\partial_\alpha F^{\alpha\beta} = j^\beta/\varepsilon_0 c, \quad \partial_\alpha *F^{\alpha\beta} = 0, \quad (1)$$

where $*F^{\alpha\beta} = (1/2)\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$. In order to get the component form of the Maxwell equations with the 3D \mathbf{E} and \mathbf{B}

$$\begin{aligned} \partial_k E_k - j^0/c\varepsilon_0 &= 0, & -\partial_0 E_i + c\varepsilon_{ijk}\partial_j B_k - j^i/c\varepsilon_0 &= 0, \\ \partial_k B_k &= 0, & c\partial_0 B_i + \varepsilon_{ijk}\partial_j E_k &= 0 \end{aligned} \quad (2)$$

from Eq. (1) one simply makes *the identification of* the six independent components of $F^{\alpha\beta}$ with six components of the 3-vectors \mathbf{E} and \mathbf{B} . These identifications are

$$E_i = F^{i0}, \quad B_i = (1/2)\varepsilon_{ijk}F_{kj} \quad (3)$$

(the indices $i, j, k, \dots = 1, 2, 3$), Eq. (11.137) in [2]. The components of the 3D fields \mathbf{E} and \mathbf{B} are written with lowered (generic) subscripts, since they are not the spatial components of the 4D quantities. This refers to the third-rank antisymmetric ε tensor too. The super- and subscripts are used only on the components of the 4D quantities. Then the 3D \mathbf{E} and \mathbf{B} , as *geometric quantities in the 3D space*, are constructed from these six independent components of $F^{\mu\nu}$ and the *unit 3D vectors* $\mathbf{i}, \mathbf{j}, \mathbf{k}$, e.g., $\mathbf{E} = F^{10}\mathbf{i} + F^{20}\mathbf{j} + F^{30}\mathbf{k}$.

It is worth noting that such an identification of the components of \mathbf{E} and \mathbf{B} with the components of $F^{\alpha\beta}$ is synchronization dependent as explicitly shown in [16]. This is also discussed in [33]. There, it is shown that the mentioned identifications are meaningless in the “r” synchronization, i.e., in the $\{r_\mu\}$ basis, in which only the Einstein synchronization is replaced by an asymmetric synchronization, the “radio” synchronization. At the end of this section, the “r” synchronization is explained in more detail. As explained in [16] (and [33]), in the “r” synchronization

$$F_r^{10} = E_1 + cB_3 - cB_2. \quad (4)$$

Hence, the identification $E_{1r} = F_r^{10}$, as in (3), shows that the component E_{1r} in the $\{r_\mu\}$ basis is expressed as the combination of E_i and B_i components from the $\{\gamma_\mu\}$ basis

$$E_{1r} = F_r^{10}, \quad E_{1r} = E_1 + cB_3 - cB_2. \quad (5)$$

This means that if the “r” synchronization is used, i.e., if the appropriate metric is used, then it is not possible to make the usual identifications (3). It follows that the usual identifications, Eq. (11.137) in [2], are meaningful *only* when the Minkowski metric, e.g., $diag(1, -1, -1, -1)$, is used. Thus, these identifications depend on the chosen synchronization, i.e., the metric. *But, different synchronizations are nothing else than different conventions and physics must not depend on conventions.*

In the usual covariant approach, e.g., [2], one transforms by the passive LT the covariant Maxwell equations (1) and finds

$$\partial'_\alpha F'^{\alpha\beta} = j'^\beta/\varepsilon_0 c, \quad \partial'_\alpha *F'^{\alpha\beta} = 0. \quad (6)$$

Under the passive LT the set of components, e.g., j^μ from the S frame transforms to j'^μ in the relatively moving inertial frame of reference S' , $j'^\mu = L^\mu_\nu j^\nu$, where, for the boost in the γ_1 direction, $L^0_0 = L^1_1 = \gamma$, $L^0_1 = L^1_0 = -\beta\gamma$, $L^2_2 = L^3_3 = 1$ and all other components are zero. Then, the same identification as in Eq. (3) is assumed to hold for the transformed components E'_i and B'_i

$$E'_i = F'^{i0}, \quad B'_i = (1/2c)\varepsilon_{ijk}F'^{kj}. \quad (7)$$

For example,

$$F'^{10} = F^{10}, \quad F'^{20} = \gamma(F^{20} - \beta F^{21}), \quad F'^{30} = \gamma(F^{30} - \beta F^{31}), \quad (8)$$

which yields (by Eqs. (3) and (7)) that

$$E'_1 = E_1, \quad E'_2 = \gamma(E_2 - \beta c B_3), \quad E'_3 = \gamma(E_3 + \beta c B_2), \quad (9)$$

see Sec. 11.10 and Eq. (11.148) in [2]. The same remark about the (generic) subscripts holds also here. Thus in the usual covariant approach the components $F^{\alpha\beta}$ are transformed by the passive LT into $F'^{\alpha\beta}$ and then it is simply argued that six independent components of $F'^{\alpha\beta}$ are the “Lorentz transformed” components E'_i and B'_i , i.e., *the LPET of the components of \mathbf{E} and \mathbf{B} are derived assuming that they transform under the LT as the components of $F^{\alpha\beta}$ transform*, Eq. (11.148) in [2]. Then, \mathbf{E}' and \mathbf{B}' are constructed in S' in the same way as in S , i.e. multiplying the components $E'_{x,y,z}$ and $B'_{x,y,z}$ by the unit 3-vectors $\mathbf{i}', \mathbf{j}', \mathbf{k}'$. This yields the LPET of the 3-vectors \mathbf{E} and \mathbf{B} , Eq. (11.149) in [2], i.e. Eq. (10) here

$$\begin{aligned} \mathbf{E}' &= \gamma(\mathbf{E} + \beta \times c \mathbf{B}) - (\gamma^2/(1 + \gamma))\beta(\beta \cdot \mathbf{E}), \\ \mathbf{B}' &= \gamma(\mathbf{B} - (1/c)\beta \times \mathbf{E}) - (\gamma^2/(1 + \gamma))\beta(\beta \cdot \mathbf{B}), \end{aligned} \quad (10)$$

where \mathbf{E}' , \mathbf{E} , β and \mathbf{B}' , \mathbf{B} are all 3-vectors.

Observe that there are no LT, or any other transformations, that transform the unit 3-vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ into the unit 3-vectors $\mathbf{i}', \mathbf{j}', \mathbf{k}'$. It is seen from Eqs. (11.148) and (11.149) in [2], i.e. from Eq. (10) here, that *the transformed \mathbf{E}' is expressed by the mixture of the 3-vectors \mathbf{E} and \mathbf{B} , and similarly for \mathbf{B}'* . The electric field \mathbf{E} in one frame is “seen” as slightly changed electric field \mathbf{E}' and an *induced magnetic field* \mathbf{B}' in a relatively moving inertial frame.

This type of the derivation of (10) was first presented in Sec. 3 in [21]. There, and in section 7.2 as well, Minkowski made the same identification of the components of $F^{\alpha\beta}$ with components of the 3-vectors \mathbf{E} and \mathbf{B} (his \mathbf{M}), as in equation (11.137) in [2]. The equations (11.148) in [2] are nothing else but the equations (6) and (7) from Sec. 3 in Minkowski’s paper [21]. Later, the same derivation is used in numerous textbooks and papers treating relativistic electrodynamics.

Minkowski’s identifications, i.e., Eq. (11.137) in [2], refer, as already stated, only to the components implicitly taken in the standard basis, which means that

they are not generally valid. Namely, in the 4D spacetime, physical quantities are represented by the abstract 4D geometric quantities that are basis independent. In some basis, as already mentioned, they are represented as CBGQs which contain *both* components *and the basis vectors* (4-vectors in the usual notation). The principle of relativity is naturally satisfied for physical laws written with such 4D geometric quantities, whereas in Einstein's formulation with the 3-vectors or with their components it has to be postulated outside the mathematical formulation of the theory. Components taken alone are only a part of a physical quantity; another, equally important, part are the basis vectors. The LPET (10) are derived using synchronization dependent identifications of *components* of $F^{\alpha\beta}$ with *components* of the 3-vectors \mathbf{E} and \mathbf{B} in both relatively moving inertial frames of reference. *This shows that both the 3-vectors \mathbf{E} and \mathbf{B} and their LPET (10) are determined in a relativistically incorrect way; the quantities entering into that derivation are not properly defined in the 4D spacetime.*

In almost all textbooks and papers on relativistic electrodynamics the LPET (10) (or (16), which are given below) are simply employed without any physical interpretation. It seems that the majority of physicists believe that it is physically justified to have, e.g., only magnetic field \mathbf{B} in one frame, in S , which transforms into a slightly changed magnetic field \mathbf{B}' and a *new electric field \mathbf{E}'* in a relatively moving S' frame. In some textbooks the authors tried to give a physical "explanation" for the appearance of that *electric field \mathbf{E}'* for the observers in S' . Thus, for example, Rosser [22] asked the reader (Problem 6.13) to interpret the origin of the electric field present in S' . Let us assume that the external magnetic field in S is due to a permanent magnet at rest in S . Then, as in Sec. 3.2 here, according to the LPET (16) ((17)) a moving magnet has an electric polarization \mathbf{P} (19), which gives an electric field outside the moving magnet. However, as discussed below, in the relativistically correct 4D geometric approach the relation (19) for the polarization \mathbf{P} , which is induced by the movement of a permanent magnetization \mathbf{M}' , does not hold. It is not derived in a relativistically correct manner; the quantities entering into that derivation are not properly defined in the 4D spacetime.

Here, for the reader's convenience, we explain here the "r" synchronization. As explained, e.g., in [16], different systems of coordinates, including different synchronizations, are allowed in an inertial frame and they are all equivalent in the description of physical phenomena. Thus in [16], both, Einstein's synchronization [1] and the "r" synchronization are exposed in detail. The "r" synchronization is commonly used in everyday life and not Einstein's synchronization. *In the "r" synchronization there is an absolute simultaneity.* Hence, contrary to the common opinion, *the relativity of simultaneity is not an intrinsic relativistic effect.* As stated in [38]: "For if we turn on the radio and set our clock by the standard announcement "...at the sound of the last tone, it will be 12 o'clock," then we have synchronized our clock with the studio clock according to the "r" synchronization. In order to treat different systems of coordinates on an equal footing it is presented, Eq. (4) in [16], the transformation matrix that connects Einstein's system of coordinates with another system of

coordinates in the same reference frame. Furthermore, Eq. (2) in [27], Eq. (1) in [16], it is derived such form of the LT, which is independent of the chosen system of coordinates, including different synchronizations. The unit vectors in the standard basis $\{\gamma_\mu\}$ and the $\{r_\mu\}$ basis, i.e., with the “r” synchronization, [16], are connected as

$$r_0 = \gamma_0, \quad r_i = \gamma_0 + \gamma_i. \quad (11)$$

Hence, the components $g_{\mu\nu,r}$ of the metric tensor are $g_{ii,r} = 0$, and all other components are $= 1$. Remember that in the $\{\gamma_\mu\}$ basis $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. (Note that in [16] and [17] the Minkowski metric is $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.) Then, according to (4) from [16], one can use $g_{\mu\nu,r}$ to find the transformation matrix R^μ_ν that connects the components from the $\{\gamma_\mu\}$ basis with the components from the $\{r_\mu\}$ basis. The only components that are different from zero are

$$R^\mu_\mu = -R^0_i = 1. \quad (12)$$

The inverse matrix $(R^\mu_\nu)^{-1}$ connects the “old” basis, $\{\gamma_\mu\}$, with the “new” one, $\{r_\mu\}$. Hence, the components of the position vector x are connected as

$$x^0_r = x^0 - x^1 - x^2 - x^3, \quad x^i_r = x^i. \quad (13)$$

Observe that vector x can be decomposed in both bases and it holds that

$$x = x^\mu \gamma_\mu = x^\mu_r r_\mu. \quad (14)$$

Obviously, the components of any vector transform in the same way as in (13), e.g., for the components of the electric field vector E it holds that

$$E^0_r = E^0 - E^1 - E^2 - E^3, \quad E^i_r = E^i. \quad (15)$$

In the same way as in (14), it can be written for the vector E that $E = E^\mu \gamma_\mu = E^\mu_r r_\mu$.

It is visible from Eqs. (11) and (13), i.e., from the fact that the metric tensor $g_{\mu\nu,r}$ is not diagonal, in the $\{r_\mu\}$ basis it is not possible to make the separation of the 4D spacetime into the time and the 3D space as it is possible in the standard basis $\{\gamma_\mu\}$. Thus *the space-time split is not possible in the $\{r_\mu\}$ basis*. In the first and the second paper (in that second paper the “r” synchronization is used as well) in [17] some of the well-known experiments: the “muon” experiment, the Michelson-Morley type experiments, the Kennedy-Thorndike type experiments and the Ives-Stilwell type experiments are analyzed using Einstein’s formulations of SR, which deals with the Lorentz contraction and the time dilation, and the ISR, i.e., the approach with 4D geometric quantities, the position vector, the distance vector between two events and the spacetime length. It is shown that all experiments are in a complete agreement, independently of the chosen synchronization, with the 4D geometric approach, i.e., with the ISR, whereas it is not the case with the Einstein’s approach with the Lorentz contraction and the time dilation if the “r” synchronization is used. In the third paper in [17] the same is shown considering in detail the Michelson-Morley experiment.

3.2 The 3-vectors \mathbf{P} and \mathbf{M} and their LPET

The LPET of the polarization and the magnetization 3-vectors \mathbf{P} and \mathbf{M} are also often derived from the covariant formulation using the mentioned, synchronization dependent, identifications of components (implicitly taken in the standard basis) of the magnetization-polarization tensor $\mathcal{M}^{\alpha\beta}$ with components of the 3-vectors \mathbf{P} and \mathbf{M} in both relatively moving inertial frames of reference. Thus, in S , these identifications are $P_i = \mathcal{M}^{i0}$, $M_i = (c/2)\varepsilon_{ijk}\mathcal{M}_{jk}$ and the same identifications hold in the relatively moving inertial frame of reference S' , $P'_i = \mathcal{M}'^{i0}$, $M'_i = (c/2)\varepsilon_{ijk}\mathcal{M}'_{jk}$, see, e.g., Secs. 18-5 and 18-6 in [23]. The same remark about the (generic) subscripts holds also here. This procedure yields

$$\begin{aligned}\mathbf{P} &= \gamma(\mathbf{P}' + \beta \times \mathbf{M}'/c) - (\gamma^2/(1+\gamma))\beta(\beta \cdot \mathbf{P}'), \\ \mathbf{M} &= \gamma(\mathbf{M}' - \beta \times c\mathbf{P}') - (\gamma^2/(1+\gamma))\beta(\beta \cdot \mathbf{M}'),\end{aligned}\quad (16)$$

see e.g., Eqs. (18-68) - (18-71) in [23], or Eqs. (4.2) in [39], or Eqs. (6.78a) and (6.81a) in [22], etc. In the mentioned equations the transformations (16) are written in an equivalent form as

$$\begin{aligned}\mathbf{P}_{\parallel} &= \mathbf{P}'_{\parallel}, & \mathbf{P}_{\perp} &= \gamma(\mathbf{P}' + \beta \times \mathbf{M}'/c)_{\perp} \\ \mathbf{M}_{\parallel} &= \mathbf{M}'_{\parallel}, & \mathbf{M}_{\perp} &= \gamma(\mathbf{M}' - c\beta \times \mathbf{P}')_{\perp}.\end{aligned}\quad (17)$$

The inverse relations are obtained in the usual way by the exchange of the primed and unprimed quantities and by the replacement $\beta \rightarrow -\beta$. The main feature of the LPET of \mathbf{P} and \mathbf{M} (16), or (17), is the same as for the LPET of \mathbf{E} and \mathbf{B} , i.e., *the components of the transformed \mathbf{P}' are expressed by the mixture of components of \mathbf{P} and \mathbf{M} , and similarly for \mathbf{M}' .*

Using completely the same procedure with the identifications of components one can derive the LPET of the 3-vectors of the electric \mathbf{p} and magnetic \mathbf{m} dipole moments from the tensor of the dipole moments $D^{a\beta}$. Hence, these LPET for the 3-vectors \mathbf{p} and \mathbf{m} are the same as (16), or (17), but with \mathbf{p} and \mathbf{m} replacing \mathbf{P} and \mathbf{M} , respectively.

The interpretation and the derivation of the transformations (16) (or (17)) in terms of simplified classical models is presented in, e.g. [23] and [22]. It is stated in Sec. 18-6 in [23] that the relation $\mathbf{P}'_{\parallel} = \mathbf{P}_{\parallel}$ is expected, “since \mathbf{P}_{\parallel} is the product of an (invariant) charge and a distance divided by a volume, *both contracted in the same ratio.*” (my emphasis) The calculation is given in section 6.7.1. in [22]. The S' frame is taken to be the rest frame of the material. The classical model assumes that in S' the dielectric consists of n' stationary dipoles/m³. If \mathbf{P}' is parallel to β , then $P' = n'p' = n'(ql_0)$, $P = np = (\gamma n')(\gamma^{-1}p') = P'$; $\gamma n'$ is *due to the contraction of the volume*, whereas $\gamma^{-1}p'$ is *due to the contraction of a distance*, $l = \gamma^{-1}l_0$. If the atomic electric dipoles are perpendicular to β , then the first term $\gamma\mathbf{P}'_{\perp}$ simply follows from $p = ql = ql_0 = p'$ (*there is no Lorentz contraction if $\mathbf{p}' \perp \beta$*), but $n = \gamma n'$, and thus the atomic electric dipoles give rise to a polarization $\gamma\mathbf{P}'_{\perp}$ in S .

It is argued, both in [23] and [22], that the extra term $(\beta \times \mathbf{M}'/c)_\perp$ in (17) has no non-relativistic counterpart. In the classical model, for purposes of calculating \mathbf{M}' , the magnetic dipoles are considered as little current loops. In section 18-4 in [23], see Fig. 18-4, and in Sec. 6.5 in [22], see Fig. 6.4 a,b, it is argued that a neutral stationary current loop, which has a magnetic moment \mathbf{m}' in its rest frame S' , acquires an electric dipole moment

$$\mathbf{p} = \beta \times \mathbf{m}'/c \quad (18)$$

if it is moving with uniform 3-velocity \mathbf{U} ($\beta = \mathbf{U}/c$) relative to the laboratory frame S . The result (18) also follows from the LPET for the 3-vectors \mathbf{p} and \mathbf{m} , which are, as already stated, the same as (16) with \mathbf{p} and \mathbf{m} replacing \mathbf{P} and \mathbf{M} , respectively. It is taken that *in the rest frame of the neutral current loop the electric moment \mathbf{p}' is zero.*

In the Ampèrian approach a permanent magnet is essentially an assembly of current loops. Hence, if (18) holds for each atomic magnetic dipole in the moving magnet, then one has for the electric dipole moment per unit volume in S , $n\mathbf{p} = \gamma n' \mathbf{U} \times \mathbf{m}'/c^2 = \gamma(\mathbf{U} \times \mathbf{M}'/c^2)_\perp$. Thus, if a permanent magnetization \mathbf{M}' is viewed from a moving frame it produces an electric polarization

$$\mathbf{P} = \gamma \mathbf{U} \times \mathbf{M}'/c^2. \quad (19)$$

In other words, *according to all usual approaches, if an observer moves with a 3-velocity \mathbf{U} relative to a medium of magnetization \mathbf{M}' that observer will observe an equivalent electric polarization given by \mathbf{P} , (19).* Adding this term to the term $\gamma \mathbf{P}'_\perp$ yields \mathbf{P}_\perp from (17).

It can be seen from the mentioned textbooks, [23], [22], that the relations (18) and (19) are obtained using *the Lorentz contraction* and *the relativity of simultaneity*.

However, as shown in [16] and in the comparison with well-known experiments that test special relativity [17], *the relativity of simultaneity, the Lorentz contraction and the time dilation are not well-defined in the 4D spacetime. They are not intrinsic relativistic effects, because they depend on the chosen synchronization.* A clear presentation of the relativistic incorrectness of the Lorentz contraction is already given in Sec. 2.2 in [27]. Already in the year 1966, Rohrlich [15] clearly explained that *the Lorentz contraction* is not a true relativistic transformation, i.e., *it has nothing to do with the Lorentz transformation.* Similarly, in the next year, Gamba [40] stated for the Lorentz contraction: “Although it is a completely useless concept in physics, it will probably continue to remain in the books as an historical relic for the fascination of the layman.” In the geometric approach, the ISR, in [16], [17], [27] it is *proved* that in the 4D spacetime two relatively moving observers cannot compare spatial lengths taken alone, which are synchronously determined for the observer. For the reader’s convenience the relativistic incorrectness of the Lorentz contraction is explicitly shown in Appendix here. Also, it is proved in [16] and [17] that it is not correct to compare the temporal distances taken alone, since they are not well-defined quantities in the 4D spacetime. The properly defined quantities are the distance

vector between two events A and B with the position vectors x_A and x_B and the spacetime length, which is a Lorentz scalar. Obviously, Gamba [40] was wrong with his statement about the Lorentz contraction. The papers [15] and [40] as well as my papers [16], [17], [27] remained almost completely overlooked and still the Lorentz contraction and the time dilation are treated in the leading physical journals and in the well-known textbooks, including [2] and [26], as intrinsic relativistic effects.

This consideration reveals in another way that the LPET of \mathbf{P} and \mathbf{M} (16), or (17), are not relativistically correct transformations, i.e., (16), *or* (17), *are not the LT but the AT*.

4. The definitions of vectors E , B and P , M in terms of F , v and \mathcal{M} , u , respectively. The Lorentz invariant field equations for vacuum and for a magnetized and polarized moving medium

Instead of dealing with quantities that are not well-defined in the 4D spacetime, like the 3-vectors \mathbf{E} and \mathbf{B} and their LPET (10), or with components implicitly taken in the standard basis as in the usual covariant approaches, [2], [26], we deal with 4D geometric quantities, which are properly defined in the 4D spacetime. Moreover, it is shown, particularly in [12], that the bivector $F = F(x)$, which represent the electromagnetic field, can be taken as the primary quantity for the whole electromagnetism and the field equation for F

$$\partial F = j/\varepsilon_0 c, \quad \partial \cdot F + \partial \wedge F = j/\varepsilon_0 c \quad (20)$$

is the basic equation. As shown in [12], the bivector field F yields the complete description of the electromagnetic field and, in fact, there is no need to introduce either the field vectors or the potentials. For the given sources the Clifford algebra formalism enables one to find in a simple way the electromagnetic field F , see Eqs. (7) and (8) in [12]. However, if one introduces the electric and magnetic fields, then they can be represented by different algebraic objects. *These fields are not determined by the usual identifications of the components, Eqs. (3) and (7), but they are derived in a mathematically correct way from F , as in Eqs. (21) and (22) here.*

In this geometric approach the electric and magnetic fields are represented by vectors $E(x)$ and $B(x)$. We deal with such representations of the electric and magnetic fields because they are simple and much closer to the classical representation of the electric and magnetic fields by the 3D vectors \mathbf{E} and \mathbf{B} than, e.g. the representations by bivectors, which are used in [18]. The decomposition of F in terms of vectors E , B and v is given as

$$F = (1/c)E \wedge v + (IB) \cdot v, \quad (21)$$

and E and B are determined as

$$E = (1/c)F \cdot v, \quad B = -(1/c^2)I(F \wedge v). \quad (22)$$

There is no rest frame for the field F , that is, for E and B , and therefore the vector v in the decomposition (21) is interpreted as the velocity vector of the observers who measure E and B fields. Then $E(x)$ and $B(x)$ are defined with respect to v , i.e., with respect to the observer. From (21) and (22) it also holds that $E \cdot v = B \cdot v = 0$; only three components of E and three components of B are independent since F is antisymmetric. The unit pseudoscalar I from (21) and (22) is defined algebraically without introducing any reference frame, as in Sec. 1.2. in the second reference in [18]. We choose I in such a way that when I is represented in the $\{\gamma_\mu\}$ basis it becomes $I = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma_5$. With such choice for I , $\{\gamma_1, \gamma_2, \gamma_3\}$ form a right-handed orthonormal set, as usual for a 3D Cartesian frame. The LT do not change the orientation of the spacetime.

The equations that correspond to equations (22) (and (21)), but in the tensor formalism, with abstract indices a, b, c, \dots , are $E^a = (1/c)F^{ab}v_b$, $B^a = (1/2c^2)\varepsilon^{abcd}F_{bc}v_d$ and $F^{ab} = (1/c)(E^a v^b - E^b v^a) + \varepsilon^{abcd}v_c B_d$, e.g., Eqs. (39) and (40) in [16]. They are based on the theorem that any second rank antisymmetric tensor can be decomposed into two vectors and a unit time-like vector (the velocity vector/ c). These equations show that in the tensor formalism too both the electric and magnetic fields can be represented by vectors.

Let us introduce the frame of “fiducial” observers as the frame in which the observers who measure fields E and B are at rest. That frame with the standard basis $\{\gamma_\mu\}$ in it is called the γ_0 -frame. In the γ_0 -frame $v = c\gamma_0$ and therefore E from (22) becomes $E = F \cdot \gamma_0$ and $B = -(1/c)\gamma_5(F \wedge \gamma_0)$. Similarly, the decomposition (21) becomes $F = E \wedge \gamma_0 + c(\gamma_5 B) \cdot \gamma_0$. All these quantities can be written as CBGQs in the standard basis $\{\gamma_\mu\}$. This yields for E and B

$$\begin{aligned} E &= E^\mu \gamma_\mu = 0\gamma_0 + F^{i0}\gamma_i, \\ B &= B^\mu \gamma_\mu = 0\gamma_0 + (1/2c)\varepsilon^{0ijk}F_{kj}\gamma_i. \end{aligned} \quad (23)$$

Note that $\gamma_0 = (\gamma_0)^\mu \gamma_\mu$ with $(\gamma_0)^\mu = (1, 0, 0, 0)$. The components of F in the $\{\gamma_\mu\}$ basis give rise to the tensor (components) $F^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot F) = (\gamma^\nu \wedge \gamma^\mu) \cdot F$. It can be easily checked that in the γ_0 -frame $E \cdot \gamma_0 = B \cdot \gamma_0 = 0$, which means that E and B are orthogonal to γ_0 . Hence, the temporal components of E and B are zero $E^0 = B^0 = 0$ and only the spatial components remain

$$E^i = F^{i0}, \quad B^i = (1/2c)\varepsilon^{0ijk}F_{kj}. \quad (24)$$

Thus, E and B actually refer to the 3D subspace orthogonal to the specific timelike direction γ_0 . In the γ_0 -frame the remaining spatial components of E and B from (24) are the same as the components of the usual 3D \mathbf{E} and \mathbf{B} , Eq. (3), which are obtained by the usual identification of the components $F^{\mu\nu}$ (implicitly taken in the standard basis) with the components of the 3D vectors \mathbf{E} and \mathbf{B} . However, there is a very important difference between the identifications (3) and the above spatial components of E and B in the γ_0 -frame, Eq. (24), which explicitly reveals that the usual procedure with the identifications (3) is not correct in the 4D spacetime. As explained above, the components of the 3D fields \mathbf{E} and \mathbf{B} in (3) are not the spatial components of the 4D quantities. They

transform according to the LPET, Eq. (11.148) in [2]. Also, the antisymmetric ε tensor is a third-rank antisymmetric tensor. On the other hand, the components of E and B in (23), i.e., in (24), are the spatial components of the 4D geometric quantities that are taken in the standard basis. They transform according to the LT, which are given below, Eq. (43). Also, the antisymmetric ε tensor in (23) is a fourth-rank antisymmetric tensor. In the usual covariant approaches one forgets about the temporal components E^0 and B^0 and simply makes the identification of six independent components of $F^{\mu\nu}$ with three components E_i and three components B_i according to the relations (3).

In Eq. (12) in [20], in the same way as in (21), the generalized magnetization-polarization bivector $\mathcal{M}(x)$ is decomposed into two vectors, the polarization vector $P(x)$ and the magnetization vector $M(x)$ and the unit time-like vector u/c

$$\mathcal{M} = P \wedge u/c + (MI) \cdot u/c^2. \quad (25)$$

There is the rest frame for a medium, i.e., for \mathcal{M} , or P and M , and therefore the vector u in the decomposition (25) is identified with bulk velocity vector of the medium in spacetime. Then, $P(x)$ and $M(x)$ are defined with respect to u as

$$P = \mathcal{M} \cdot u/c, \quad M = cI(\mathcal{M} \wedge u/c) \quad (26)$$

and it holds that $P \cdot u = M \cdot u = 0$; only three components of P and three components of M are independent since \mathcal{M} is antisymmetric. As in the case with F in (21) and (22), it is visible from (26) that P and M depend not only on \mathcal{M} but on u as well.

Here, we briefly examine the Lorentz invariant field equations for vacuum and for a magnetized and polarized moving medium. Inserting (21) into the field equation for F (20) one finds the field equation in terms of E and B ;

$$\partial(E \wedge (v/c) + (IB) \cdot v) = j/\varepsilon_0 c. \quad (27)$$

As explained in [8], Eq. (27) represents the Lorentz invariant generalization of the usual Maxwell equations. That form (27) is the most general form of the field equations with electric and magnetic fields as properly defined quantities in the 4D spacetime.

The equation (27) with the geometric product can be divided into the vector part (with sources)

$$\partial \cdot (E \wedge v/c + (IB) \cdot v) = j/\varepsilon_0 c \quad (28)$$

and the trivector part (without sources)

$$\partial \wedge (E \wedge v/c + (IB) \cdot v) = 0. \quad (29)$$

In that form it is clear that *it is not possible to separate the field equation with sources for the E field from that one for the B field. Thus, in the 4D spacetime, the generalizations with 4D geometric quantities of the usual Ampère-Maxwell law and Gauss's law are inseparably connected in one law - Eq. (28). Similarly, in Eq. (29), Faraday's law and the law that expresses the absence of magnetic*

charge are also inseparably connected in one law, which is expressed in terms of the 4D geometric quantities. This is an essential difference relative to Maxwell's equations with the 3-vectors \mathbf{E} and \mathbf{B} .

The mathematical reason for such an inseparability is that, e.g., the gradient operator ∂ is a vector field defined on the 4D spacetime. If represented in some basis then its vector character remains unchanged only when *all its components together with associated basis vectors* are taken into account in the considered equation. The same holds for other vectors E , B , j , etc. and multivectors like F , \mathcal{M} , For example, in general, in the 4D spacetime, the current density vector j is a well-defined physical quantity, but it is not the case with the usual charge density ρ and the usual current density \mathbf{j} as a 3-vector. Similarly, in general, the gradient operator ∂ cannot be divided into the usual time derivative and the spatial derivatives, e.g., in the $\{\gamma_\mu\}$ basis with the “r” synchronization. In the 4D spacetime, an independent physical reality is attributed to the position vector x , the gradient operator ∂ , the current density vector j , the vectors of the electric and magnetic fields E and B , respectively, etc., but not to the 3-vector \mathbf{r} and the time t , to the 3-vectors \mathbf{j} , \mathbf{E} , \mathbf{B} , etc.

The generalization of (20) to a moving medium is presented in [20] and it is obtained simply replacing F by $F + \mathcal{M}/\varepsilon_0$, which yields the primary equations for the electromagnetism in moving media

$$\partial(\varepsilon_0 F + \mathcal{M}) = j^{(C)}/c; \quad \partial \cdot (\varepsilon_0 F + \mathcal{M}) = j^{(C)}/c, \quad \partial \wedge F = 0, \quad (30)$$

where $j^{(C)}$ is the conduction current density of the *free* charges and $j^{(M)} = -c\partial \cdot \mathcal{M}$ is the magnetization-polarization current density of the *bound* charges. The total current density vector j is $j = j^{(C)} + j^{(M)}$. Inserting the decomposition (25) into $j^{(M)}$ it follows that

$$j^{(M)} = -(\partial \cdot P)u + (u \cdot \partial)P + (1/c)[u \wedge (\partial \wedge M)]I. \quad (31)$$

In the standard basis $\{\gamma_\mu\}$ and in the rest frame of the medium, $u = c\gamma_0$, $P^0 = M^0 = 0$,

$$j^{(M)\mu} = (-c\partial_k P^k, c\partial_0 P^i - \varepsilon^{0ijk}\partial_j M_k). \quad (32)$$

In the usual formulation with the 3-vectors these components correspond to, e.g., Eq. (18-61) in [23], i.e., $\rho = -\nabla \cdot \mathbf{P}$, $\mathbf{j} = \partial \mathbf{P}/\partial t + \nabla \times \mathbf{M}$. In (32), the components P^i , M^i with the upper indices correspond to the components of the 3-vectors. However, *in the 4D spacetime it is not correct to write the components of the properly defined vectors in terms of the 3-vectors and operations with them. There are no 3-vectors in the 4D spacetime.*

In most materials \mathcal{M} is a function of the field F and this dependence is determined by the constitutive relations. In that case (30) are well-defined equations for F . Recently, the constitutive relations and the magnetoelectric effect for moving media are investigated in detail in [41].

Then, in [20], the general form of the field equation for a magnetized and polarized moving medium expressed in terms of $E(x)$, $B(x)$, $P(x)$ and $M(x)$ is obtained by the insertion of Eqs. (21) and (25) into the field equation (30). It

is Eq. (15) in [20], which is the generalization of Eq. (27) (with the geometric product) to the moving media. The generalizations of Eq. (28) to the moving media is given by Eq. (16) in [20], the vector part (with sources), i.e., in the “source representation” by the equation

$$\partial \cdot \{\varepsilon_0[E \wedge v/c + (IB) \cdot v]\} = j^{(C)}/c - \partial \cdot [P \wedge u/c + (1/c^2)(MI) \cdot u], \quad (33)$$

according to which the sources of E and B fields are $j^{(C)}$ and P and M . Obviously, from (33), *it is not possible to separate the field equation with sources for the E field from that one for the B field.* This is an essential difference relative to Maxwell’s equations with the 3-vectors \mathbf{E} , \mathbf{B} , \mathbf{P} and \mathbf{M} . The field equation without sources, the trivector part, remains unchanged relative to the corresponding equation for vacuum (29)

$$\partial \wedge [E \wedge v/c + (IB) \cdot v] = 0. \quad (34)$$

In Eq. (33), i.e., in Eq. (18) in [20], there are two different velocities u and v and such an equation is not previously reported in the physics literature.

As stated in [20], Eq. (30), i.e., Eqs. (33) and (34) *comprise and generalize all usual Maxwell’s equations (with 3-vectors) for moving media.*

Again, as in the above discussion for the vacuum, *in the 4D spacetime*, in contrast to the usual formulation of electromagnetism with the 3-vectors \mathbf{E} , \mathbf{B} , \mathbf{P} , \mathbf{M} , \mathbf{j} , ... , there are no two laws, the Ampère-Maxwell law and Gauss’s law, but *only one law*, that is expressed by Eq. (33) and the same for other two laws and Eq. (34).

In the same way as in [12] one can derive the expression for the Lorentz force density k_L ,

$$k_L = F \cdot j/c = (1/c)F \cdot (j^{(C)} - c\partial \cdot \mathcal{M}). \quad (35)$$

Inserting the decompositions (21) and (25), i.e., (21) and (31), into k_L (35) one can find k_L expressed in terms of $E(x)$, $B(x)$, $P(x)$ and $M(x)$. This is discussed in Sec. 9 in connection with the “charge-magnet paradox.”

5. The LT of vectors E and B ; both F and the observer are transformed

As seen from (21) and (22) all quantities F , E , B and v are abstract 4D geometric quantities. If these geometric quantities from (21) and (22) are represented in some basis then they contain both components and basis vectors. In his fundamental work, Minkowski, in Sec. 11.6 in [21], wrote the relation (55) that corresponds to (21), but he considered that the quantities w , Φ and Ψ , which correspond to our v , E and B , are 1×4 matrices and that F is a 4×4 matrix. Their components are implicitly determined in the standard basis. In Sec. 11.6 in [21], the next paragraph below Eq. (44), Minkowski described how w and F separately transform under the LT A (the matrix of the LT is denoted as A in [21]) and then how the product wF transforms. Thus, he wrote

$$w' = wA \quad (36)$$

for the LT of the velocity vector w and

$$F' = A^{-1}FA \quad (37)$$

for the LT of the field-strength tensor. Then, the mathematically correct LT of $\Phi = wF$ are

$$\Phi = wF \longrightarrow \Phi' = (wA)(A^{-1}FA) = (wF)A = \Phi A, \quad (38)$$

which means that under the LT both quantities, the velocity w and F are transformed and their product transforms as any other vector (i.e., in [21], as an 1×4 matrix) transforms. The most important thing is that *the electric field vector Φ transforms by the LT again to the electric field vector Φ' ; there is no mixing with the magnetic field Ψ .*

These correct LT of the electric and magnetic fields are reinvented and generalized in terms of 4D geometric quantities in [6-11]. In the geometric algebra, the LT (the active ones) are described by rotors R , $R\tilde{R} = 1$, where the reverse \tilde{R} is defined by the operation of reversion according to which $\widetilde{AB} = \tilde{B}\tilde{A}$, for any multivectors A and B , $\tilde{a} = a$, for any vector a , and it reverses the order of vectors in any given expression. For boosts in an arbitrary direction the rotor R is given by Eq. (8) in [7, 9], or Eq. (10) in [11], as

$$R = (1 + \gamma + \gamma\gamma_0\beta)/(2(1 + \gamma))^{1/2}, \quad (39)$$

where $\gamma = (1 - \beta^2)^{-1/2}$, the vector β is $\beta = \beta n$, β on the r.h.s. of that equation is the scalar velocity in units of c and n is not the basis vector but any unit space-like vector orthogonal to γ_0 . Then, *any multivector N transforms by active LT in the same way*, i.e., as

$$N \rightarrow N' = RN\tilde{R}. \quad (40)$$

Hence, vector E transforms by the LT R as $E \longrightarrow E' = RE\tilde{R}$. In the γ_0 -frame, $v = c\gamma_0$ is taken in (22). Then E becomes $E = F \cdot \gamma_0$ and it transforms under the LT in the same manner as in Minkowski's relation (38), i.e., that both F and v are transformed by the LT R as

$$E = F \cdot \gamma_0 \longrightarrow E' = R(F \cdot \gamma_0)\tilde{R} = (RF\tilde{R}) \cdot (R\gamma_0\tilde{R}). \quad (41)$$

These correct LT give that

$$E' = E + \gamma(E \cdot \beta)\{\gamma_0 - (\gamma/(1 + \gamma))\beta\}. \quad (42)$$

In the same way vector B transforms and vectors P , M as well, but for P and M the LT, like (42), are the transformations from the rest frame of the medium ($u = c\gamma_0$). For boosts in the direction γ_1 one has to take that $\beta = \beta\gamma_1$ (on the l.h.s. is vector β and on the r.h.s. β is a scalar) in the above expression for the rotor R (all in the standard basis). Hence, in the $\{\gamma_\mu\}$ basis and when $\beta = \beta\gamma_1$ Eq. (42) becomes

$$E'^\nu\gamma_\nu = -\beta\gamma E^1\gamma_0 + \gamma E^1\gamma_1 + E^2\gamma_2 + E^3\gamma_3. \quad (43)$$

As already mentioned in Sec. 1, the relations (42) and (43) are the fundamental results, which show that *under the relativistically correct LT the electric field vector E transforms again to the electric field vector E' ; there is no mixing with the magnetic field B* . The same happens with vectors P and M . The same fundamental result can be obtained if electric and magnetic fields are represented, e.g., by bivectors as in [9]. In general, it can be stated that *the LT always transform the 4D algebraic object (vector, bivector) representing the electric field only to the electric field, and similarly for the magnetic field*.

It is important to note that E' (and B') from (42) and (43) are not orthogonal to γ_0 , i.e., they have temporal components $\neq 0$. They do not belong to the same 3D subspace as E and B , but they are in the 4D spacetime spanned by the whole standard basis $\{\gamma_\mu\}$.

The same components as in (43) would be obtained for $\Phi' = \Phi A$ in Minkowski's relation (38) if the components of w are $(0, 0, 0, ic)$ in his notation, which corresponds to $v = c\gamma_0$ in our formulation. It is worth noting that only in Sec. 11.6 in [21] Minkowski dealt with vectors (only components) w , Φ and Ψ , but in the rest of [21] he exclusively dealt with the usual 3-vectors \mathbf{v} , \mathbf{E} and \mathbf{B} (our notation) and not with correctly defined vectors w , Φ and Ψ .

In Sec. 11 under the title “Minkowski in 1908, and Ivezić Since 2003: Lorentz Covariance” in the third paper in [19] Oziewicz, from the mathematical point of view, nicely explains the results obtained in my papers [6-11]. (The references in the quoted part refer to the mentioned Oziewicz's paper.) He states:

“Ivezić observed the logical and mathematical inconsistency of textbook treatments of the Lorentz-covariance since 2003. He noted that it is illogical to consider a closed differential biform F to be Lorentz-covariant, and at the same time, keep observer's time-like vector field, a ‘4-velocity’, $P \simeq (1, 0, 0, 0)$, to be Lorentz-invariant-absolute. For example, compare how an absolute observer is hidden in calculations presented in (Misner, Thorne & Wheeler [31], Chapt. 3).

Minkowski [1], and then Ivezić [7-10], observed correctly that if a Lorentz transformation is an isomorphism of a vector space, then the entire algebra of tensor fields must be Lorentz-covariant. Every vector is Lorentz-covariant, and an observer-monad timelike vector field, also must be Lorentz-covariant. All tensor fields, F and P , must be Lorentz-covariant. An active Lorentz transformation must act on all tensor fields, including an observer's time-like vector field. Hence electromagnetic field F , potential A , and Paul P , must be Lorentz-covariant (Ivezić [7-10]).

Instead of Fock's and Jackson's transformations (10.2) - (10.3), (our Eqs. (44) and (45), my remark) Ivezić defined the Lorentz-covariance for the compound electric and magnetic fields, (7.2), (it corresponds to our Eq. (22), my remark) exactly as defined by Minkowski in [1], §11.6, just before formula (46). We stress that Minkowski in [1] does not in practice use his definition of Lorentz-covariance. Instead of (10.2) - (10.3), the Lorentz transformation of electric and magnetic concomitant vector fields according to the Minkowski and Ivezić definition of Lorentz covariance is: ...” given by Oziewicz's equations (11.4) - (11.9). His relation (11.5) is our Eq. (42).

Here, it is at place to give an interesting remark regarding Oziewicz's papers [19]. It has often been argued that it cannot be that the 3-vectors \mathbf{E} , \mathbf{B} , \mathbf{P} and \mathbf{M} and their transformations (10) and (16), or (17), have to be replaced by the 4D geometric quantities, e.g., vectors E , B , P and M and by their mathematically and relativistically correct LT (42) and (43). On the other hand Oziewicz, in difference to all others, correctly considers from the outset that *there are no 3-vectors in the 4D spacetime*. But, he incorrectly considers that, e.g., the transformations of the 3-vectors \mathbf{E} and \mathbf{B} , Eq. (10), i.e., Jackson's Eq. (11.149) in [2], are his equations (10.2) - (10.3), i.e., our Eqs. (44) and (45). His equations (10.2) - (10.3) are the equations with 4D geometric quantities and they correspond to our Eqs. (44) and (45) given below, whereas Eq. (10) contains *only the 3-vectors \mathbf{E} , \mathbf{B} , \mathbf{E}' , \mathbf{B}' and the velocity 3-vector*. Thus, his equations (10.2) - (10.3) are not "Fock's and Jackson's transformations."

There is a very important consequence of the LT (42) and (43), or the same for P and M . As mentioned above, *under the relativistically correct LT the polarization vector P transforms again to the polarization vector P' ; there is no mixing with the magnetization vector M* . On the other hand, according to (10) if there is an external, static, magnetic field (3-vector) outside, e.g., a stationary current loop then there is the magnetic field *and a static electric field (3-vector)* outside the same current loop which moves with uniform 3-velocity \mathbf{U} .

According to the LT (42) and (43), if there is an electric field outside *moving* magnet *it would necessary need to exist outside the same but stationary magnet*. Thus, in the Ampèrian approach, there is a polarization vector P (remember that P is a 4D geometric quantity and not a 3-vector) for a stationary permanent magnet as an assembly of small current loops. As explained in Sec. 7.1 below, this happens because every current loop behaves like an electric dipole at points far from that loop. Then, that P induces an external electric field. This will be discussed in much more detail in Sec. 8 below.

6. The usual transformations of vectors E and B ; only F is transformed but not the observer

Let us examine what will be obtained if in the transformation of $E = F \cdot \gamma_0$ only F is transformed by the LT R , Eq. (39), but not the velocity of the observer $v = c\gamma_0$. Of course, it will not be the LT of $E = F \cdot \gamma_0$, because they are given by Eq. (41). Thus

$$E = F \cdot \gamma_0 \longrightarrow E'_F = (RF\tilde{R}) \cdot \gamma_0 = F' \cdot \gamma_0. \quad (44)$$

This procedure yields that

$$E'_F = \gamma\{E + (\beta \wedge \gamma_0 \wedge cB)I\} + (\gamma^2/(1 + \gamma))\beta(\beta \cdot E). \quad (45)$$

If Eqs. (44) and (45) are written in the standard basis and if it is taken that $\beta = \beta\gamma_1$, then they become

$$\begin{aligned} E'_F &= F' \cdot \gamma_0 = 0\gamma_0 + F'^{i0}\gamma_i = E'_F{}^\nu\gamma_\nu \\ &= E^1\gamma_1 + \gamma(E^2 - c\beta B^3)\gamma_2 + \gamma(E^3 + c\beta B^2)\gamma_3. \end{aligned} \quad (46)$$

Similarly, we find for B'_F

$$\begin{aligned} B'_F &= -(1/c)\gamma_5(F' \wedge \gamma_0) = 0\gamma_0 + (1/2c)\varepsilon^{0ijk}F'_{kj}\gamma_i = B'^{\nu}_F\gamma_\nu \\ &= B^1\gamma_1 + (\gamma B^2 + \beta\gamma E^3/c)\gamma_2 + (\gamma B^3 - \beta\gamma E^2/c)\gamma_3. \end{aligned} \quad (47)$$

From the transformations (46) and (47) one simply finds the transformations of the spatial components E'^i_F and B'^i_F

$$E'^i_F = F'^{i0}, \quad B'^i_F = (1/2c)\varepsilon^{0ijk}F'_{kj}, \quad (48)$$

which is the relation (24) but with the primed quantities.

It is seen from (46) that *the components of the transformed E'_F are expressed by the mixture of components of E and B . The same conclusion follows for B'_F from (47).*

The transformation (45) can be compared with the LPET for the 3-vector \mathbf{E} that are given by the first equation in (10), and Eq. (46) can be compared with Eq. (11.148) in [2], i.e., with Eq. (9) here. Remember that in Eq. (10) \mathbf{E}' , \mathbf{E} , \mathbf{B}' , \mathbf{B} and β are all the usual 3-vectors. The comparison of Eq. (46) with Eq. (11.148) in [2] shows that *the transformations of components (taken in the standard basis) of E'_F are exactly the same as the transformations of $E_{x,y,z}$ from Eq. (11.148) in [2].* The same conclusion holds for the comparison of Eq. (47) and $B_{x,y,z}$ from Eq. (11.148) in [2]. The result that the components in (46) are the same as the components of \mathbf{E}' from (10) is completely understandable. Namely, (45) and (46) are obtained by the application of the LT *only* to F . On the other hand, it is already stated in Sec. 3.1 that the LPET of the components of \mathbf{E} and \mathbf{B} are derived assuming that they transform under the LT as the components of $F^{\alpha\beta}$ transform, Eqs. (3) and (7).

In contrast to the LT of E (43) (and the same for B), it is visible from (46), (47) and (48) that *E'_F and B'_F are again in the 3D subspace of the γ_0 -observer, as it holds for E and B in the γ_0 -frame, Eqs. (23) and (24). Thus for the transformed E'_F and B'_F again hold that $E'^0_F = B'^0_F = 0$, i.e., that $E'_F \cdot \gamma_0 = B'_F \cdot \gamma_0 = 0$ as for E and B in the γ_0 -frame. This shows in another way that *the LPET (46), (47) and (48) are not the LT, since the LT cannot transform some quantity from the 3D subspace again only to the 3D subspace.**

The transformations (44) and (46)-(48) are first discussed in detail in [6-11] and compared with the LPET (11.148) and (11.149) from [2], whereas the general form of E'_F , Eq. (45), is first given in [19].

We now point out another difference between the LT and the LPET. If instead of the active LT we consider the passive LT then, e.g. the vector $E = E^\nu\gamma_\nu = E'^\nu\gamma'_\nu$ will remain unchanged, because the components E^ν transform by the LT and the basis vectors γ_ν by the inverse LT leaving the whole E invariant under the passive LT. Of course, the same holds for all bases including those with nonstandard synchronizations, as discussed, e.g., in [16] and [33]. For the $\{r_\mu\}$ basis, this can be easily proved using R^μ_ν , i.e., Eqs. (11) and (15);

$$E = E^\nu\gamma_\nu = E'^\nu\gamma'_\nu = E^\nu_r r_\nu = E'^\nu_r r'_\nu. \quad (49)$$

The primed quantities in both bases are the Lorentz transforms of the unprimed ones. For the general form of the LT, that is independent of the chosen system of coordinates, including different synchronizations, see, e.g., Eq. (1) in [16]. The LT in the $\{r_\mu\}$ basis are given in the same paper by Eq. (2) or Eq. (21) in [33].

This invariance of E means that the electric field E is the same physical quantity for all relatively moving observers and for all bases used by them. In the same way this requirement has to be fulfilled for any well-defined 4D quantity.

It is not so with the 3-vector \mathbf{E} and its LPET, or, equivalently, with E'_F . Namely, $\mathbf{E} = E_x \mathbf{i} + E_y \mathbf{j} + E_z \mathbf{k}$ is completely different than \mathbf{E}' , $\mathbf{E}' = E'_x \mathbf{i}' + E'_y \mathbf{j}' + E'_z \mathbf{k}'$ from (10), $\mathbf{E} \neq \mathbf{E}'$, and the same holds for E'_F , i.e., $E^\nu \gamma_\nu \neq E'^\nu \gamma'_\nu$. This means that although \mathbf{E} and \mathbf{E}' are measured by different observers *they are not the same quantity for such relatively moving observers*. The observers are not looking at the same physical object, here the electric field vector, but at two different objects. *Every observer makes measurement of its own 3-vector field, \mathbf{E} and \mathbf{E}' , and such measurements are not related by the LT.* As far as relativity is concerned the quantities, e.g., \mathbf{E} and \mathbf{E}' , i.e., $E^\nu \gamma_\nu$ and $E'^\nu \gamma'_\nu$, are not related to one another. Their identification is a typical case of *mistaken identity*.

In the 4D geometric approach, i.e., in the ISR, different relatively moving inertial 4D observers can compare only 4D quantities, here $E^\nu \gamma_\nu$ and $E'^\nu \gamma'_\nu$, because they are connected by the LT. The experimentalists have to measure *all components* of 4D quantities, here of E , in both frames S' and S . The observers in S' and S are able to compare only such complete set of data which corresponds to the *same* 4D geometric quantity. Hence, from the point of view of the ISR the transformations for E'^i_F and B'^i_F , Eq. (48), i.e., Eq. (11.148) in [2], are not the LT. Therefore, contrary to the general belief, *it is not true from the geometric approach viewpoint that*, e.g., Sec. 11.10 in [2]: “A purely electric or magnetic field in one coordinate system will appear as a mixture of electric and magnetic fields in another coordinate frame.”

7. Clausius' hypothesis and the second-order electric field outside a stationary superconductor with steady current

7.1 The second-order electric field outside a stationary superconductor with steady current

As stated in Sec. 3.2, in the classical model, the magnetic dipoles are considered as little current loops. In Sec. 18-4 in [23] and Sec. 6.5. in [22], and in all other usual approaches, it is assumed that a stationary current loop *with steady current* is globally and *locally* charge neutral (Clausius' hypothesis, (1877)), i.e., *it is simply supposed* that in the ions' rest frame S the charge density of the moving electrons is $\rho_- = -\rho_0$, where ρ_0 is the positive charge density for the wire at rest but without a current. Clausius (see Ref. [4] in the first paper in [28]) stated that hypothesis in another but equivalent way, i.e., he stated, as an “experimental assumption” that a “closed current in a stationary conductor exerts no force on stationary electricity.” But, it is also assumed that the

same current loop does not remain locally neutral when observed from another inertial frame. Namely, in [23], [22], ..., it is argued that the legs parallel to the 3-velocity \mathbf{U} will carry the charges equal in magnitude but opposite in sign, because they are Lorentz contracted, whereas the legs perpendicular to \mathbf{U} remain uncharged, because there is no Lorentz contraction for them. Thus, an electric dipole moment given by the expression (18) is obtained. Hence, there is an external, static, magnetic field (3-vector) outside the stationary current loop, but there is the magnetic field *and a static electric field (3-vector)* outside the same current loop which moves with uniform 3-velocity \mathbf{U} . Note that the legs of the current loop are treated as that they are infinite wires with steady currents. It can be seen from the mentioned textbooks, [23], [22], that such result is obtained using *the Lorentz contraction* and *the relativity of simultaneity*. In all these derivations it is also used the conventional definition of charge in terms of 3D quantities,

$$Q = (1/c) \int_{V(t)} j^0(\mathbf{r}, t) dV. \quad (50)$$

In that definition the volume $V(t)$ is taken at a particular coordinate time t and it is stationary in some inertial frame of reference S . The values of the charge density $\rho(\mathbf{r}, t) = j^0(\mathbf{r}, t)/c$ are taken simultaneously for all \mathbf{r} in $V(t)$. It is supposed in all usual treatments (see, particularly, the well-known Purcell's textbook [24]) that the volume elements dV' are Lorentz "contracted" in a relatively moving inertial frame of reference S' and all of them, i.e., the whole volume $V'(t')$, are taken simultaneously at some arbitrary t' in S' . *The coordinate time t' in S' is not connected in any way with t in S .* Also, it is assumed that j^0 from S is transformed (using the Lorentz "contraction") only to j'^0 in S' and all j'^0 are taken simultaneously at the same t' in S' . The new

$$Q' = (1/c) \int_{V'(t')} j'^0(\mathbf{r}', t') dV' \quad (51)$$

in S' is considered to be equal to the charge Q in S , $Q' = Q$, (the total charge is invariant). But we remark that the charge Q defined in such a manner cannot be invariant under the LT. The LT cannot transform one component j^0 from an inertial frame of reference S to the same component j'^0 in S' . Also, if all j^0 values are taken simultaneously at some t in S then the LT cannot transform them to the values j'^0 which are again all simultaneous, but now *at some arbitrary t' in S'* . This consideration shows that such usual definition of charge cannot be the relativistically correct definition.

In the 4D spacetime, two relatively moving observers cannot compare spatial lengths taken alone, which are synchronously determined for the observer, see Appendix here. Similarly, in the 4D spacetime, the temporal distances taken alone are not well-defined and two relatively moving observers cannot compare them. For the thorough discussion of these usual definitions of the Lorentz contraction and the time dilation see, e.g., Secs. 4. - 4.2 and Figs. 3 and 4 in [16]. Also, a clear presentation of the relativistic incorrectness of the Lorentz

contraction is already given in Sec. 2.2 in [27]. In the ISR, the properly defined quantity is the distance vector and the spacetime length, which is a Lorentz scalar, see, e.g., Secs. 3 - 3.2 and Figs. 1 and 2 in [16] and also Sec. 2.1 in [27].

Furthermore, the above mentioned conventional definition of charge in terms of 3D quantities is objected in Sec. 3 in [27] and Sec. 5.3 in [16]. There, it is shown that such definition with 3D quantities has to be replaced by the definition in terms of 4D geometric quantities, i.e., the charge is defined as a Lorentz scalar. The total electric charge Q in a three-dimensional hypersurface H (with two-dimensional boundary δH), as a Lorentz scalar, is defined by the equation

$$Q_{\delta H} = (1/c) \int_H j \cdot n dH, \quad (52)$$

where j is the current density vector and the vector n is the unit normal to H .

In Secs. 3 and 3.1 in [27], the external electric field for an infinite wire with a steady current and the Clausius hypothesis are examined in detail.

In the prerelativistic physics and in Einstein's formulation of the relativistic physics the charge density is well-defined quantity both for charges at rest and for the moving charges. As discussed in Sec. 3 in [27], it is considered in the usual approaches, e.g., [24-26], that the charge density of the moving charges is properly defined; it is enhanced by $\gamma = (1 - \beta^2)^{-1/2}$ relative to the proper charge density due to the *Lorentz contraction of the moving volume*. Therefore, both in the prerelativistic physics and in Einstein's formulation of the relativistic physics the Clausius hypothesis is meaningful, i.e., it can be properly formulated. Hence, in the usual approaches, *there is no external electric field for a stationary current-carrying conductor*. Of course, the whole consideration refers to an ideal conductor or to a superconductor, because for a stationary *resistive* conductor carrying constant current there is always an external static magnetic field *and a time independent external electric field that is proportional to the current and which is caused by the distribution of surface charges on the conductor*, see, e.g., [42] and references therein.

On the other hand, as already mentioned several times, see, e.g., Sec. 3. in [27] and Sec. 4 here, in the 4D spacetime, only the current density vector j is a well-defined physical quantity, but not the usual charge density ρ and the usual current density \mathbf{j} as a 3-vector. *In the 4D spacetime it is not possible to give a definite physical meaning to the charge density of moving charges*. As discussed in Appendix, the Lorentz contracted length is meaningless in the 4D spacetime. As shown in Secs. 3 - 3.3 in [27], for an infinite wire with a steady current (the wire is situated along the x^1 axis) and if the standard basis is introduced, one can take that in S' , in which the drift velocity 3-vector of the electrons is zero, the current density vector of the electrons (in one spatial dimension) is

$$j'_- = (-c\rho_0)\gamma_0 + 0\gamma_1, \quad (53)$$

i.e., as that the proper charge density ρ'_- of the electrons ($j'^i_- = 0$) is equal to $-\rho_0$. This is completely different than the Clausius hypothesis. Then, by means of (53) and the LT one finds the current density vectors in S , the rest

frame of the wire, i.e., the lab frame, as (only components)

$$j_-^\mu = (-c\gamma\rho_0, -c\gamma\beta\rho_0), \quad j^\mu = (c(1-\gamma)\rho_0, -c\gamma\beta\rho_0), \quad (54)$$

where $j = j^\mu\gamma_\mu$ is the total current density vector in S , i.e., (components in the standard basis) $j^\mu = j_-^\mu + j_+^\mu$ and $j_+^\mu = (c\rho_0, 0)$. Observe that it holds that $j = j^\mu\gamma_\mu = j'^\mu\gamma'_\mu$, where the primed quantities are the Lorentz transforms of the unprimed ones. The equations (53) and (54) are Eqs. (11) and (12) in [27], respectively.

The same equations, (53) (i.e., only components) and (54) were already obtained in [32]. There, in contrast to the usual approach, the Lorentz contraction is introduced not only for the mean spacing between moving ions in the S' frame, but also it is assumed that there is a Lorentz contraction of the mean spacing between moving electrons in the lab frame, i.e., in the stationary wire with steady current. This may seem surprising that the same equations exist in [32] in which the Lorentz contraction is used and in [27] and here, where the 4D geometric quantities are used. But, the results obtained in [32] are not actually based on the use of the Lorentz contraction, than on the *assumption* that in the electrons' rest frame S' the electrons' charge density ρ'_- is $= -\rho_0$. The Lorentz contraction of the mean distance between moving electrons was only taken as the interpretation for the assumption that $\rho'_- = -\rho_0$. In the 4D geometric approach from [27] and here Eq. (53) is neither *hypothesis* as in the traditional approach with the 3D quantities and the Lorentz contraction, nor the *assumption* as in [32], but it is a consequence of the covariant definition of an invariant charge (52) and of the invariance of the rest length (see Appendix), i.e., it is the result of the use of the correctly defined 4D geometric quantities.

The components in the standard basis of the electric and magnetic fields are determined from the known current density vector j^μ (54) in Secs. 3.2 - 3.3 in [27]. Taking that the rest frame of the wire is the γ_0 -frame, i.e., that $E^0 = 0$, $E^i = F^{i0}$, then the external electric field is $E = E^\mu\gamma_\mu = 0\gamma_0 + E^i\gamma_i$, where the components are given by Eq. (22) in [27]; that field is in the plane orthogonal to the wire and in the radial direction in that plane (the wire is situated along the x^1 axis)

$$E^0 = E^1 = 0, \quad E^2 = 2k(1-\gamma)\rho_0 ya^{-2}, \quad E^3 = 2k(1-\gamma)\rho_0 za^{-2}, \quad (55)$$

where $k = 1/4\pi\epsilon_0$, $a^2 = y^2 + z^2$, $(x^1, x^2, x^3) = (x, y, z)$. Then, it is concluded in Sec. 3.3. in [27]: “The equation (55) shows that the observer who is at rest relative to a wire with steady current will see, i.e., measure, the second-order electric field outside such a current-carrying conductor.” (“the second-order electric field” means that $E^i \propto U^2/c^2$, where U is the magnitude of the drift velocity 3-vector of the electrons.) Note that such fields, but as the 3-vectors, are first predicted on different grounds in [32], see the above discussion about the assumption from [32] that $\rho'_- = -\rho_0$. As already mentioned, the second-order electric field (55) exists in a resistive wire with a constant current as well, but there it is much smaller but the contribution to the external electric field that is caused by the quasistatic surface charges.

Recently, the same treatment with the Lorentz contraction and the same results as in [32] are presented in [31]. That work, [31], from the theoretical point of view is almost the same as the treatment in [32], i.e., it is not with 4D geometric quantities and thus it is not a mathematically and relativistically correct treatment. Several results from [32] are incorrectly understood and interpreted in [31]. This will not be discussed here since both papers deal with the Lorentz contraction. But [31] is an important progress in the investigation of the existence of the second-order electric field, because it presents, as asserted in [31]: “a new analysis of the experimental sensitivity required to observe the hypothesized effect and analyzes the feasibility of several novel experimental methods to make such an observation.” This will be discussed below in Sec. 7.2 together with the discussion of the already performed experiments and some suggested experiments.

Under the passive LT it holds that $E = E^\mu \gamma_\mu = E'^\mu \gamma'_\mu$. This essential feature of the approach with 4D geometric quantities, i.e., of the ISR, shows that *if the electric field vector exists in one inertial frame of reference, say in the rest frame of the electrons, the S' frame, as in all usual approaches, then it must necessarily exist in the rest frame of the ions, i.e., of the wire, the S frame.*

Similarly, it is obtained in [27] that the components in the standard basis of the magnetic field are $B^0 = 0$ and B^i , which are the same as for the usual expression for the magnetic field of an infinite straight wire with current (only the components j^i are γ times bigger).

$$B^0 = B^1 = 0, \quad B^2 = \gamma \mu_0 I y a^{-2}, \quad B^3 = \gamma \mu_0 I z a^{-2}, \quad (56)$$

where $I = \rho_0 A U$, A is the cross-sectional area. The vectors of the electric and magnetic fields in some relatively moving frame, e.g., the rest frame of the electrons, can be obtained using (55), (56) and the LT (42) and (43). Again, as for E , it holds that under the passive LT B is unchanged, $B = B^\mu \gamma_\mu = B'^\mu \gamma'_\mu$, as can be easily checked.

In Sec. 4 in [27], the same consideration is presented for a current loop and it is shown that *the second-order external electric field exists not only for a moving current loop, as in the usual approaches, but for the stationary current loop as well.* There are opposite charges on opposite sides of a square loop with current, but the total charge of that loop is zero. All these charges are invariant charges, which means that they are the same for both, moving and stationary current loop. They are defined as the Lorentz scalars, i.e., as in Eq. (52) for $Q_{\delta H}$. *At points far from that current loop it behaves like an electric dipole, but as a 4D geometric quantity.* It is incorrectly asserted in the published paper in [27] that such a distribution of charges behaves like an electric quadrupole. This is corrected in the v2 of the second paper in [27]. However, it is worth noting that, in the same way as in the usual approaches, the legs of the current loop are treated as that they are infinite wires with steady currents.

Thus, in the 4D geometric approach, i.e., in the ISR, the relation (18), which is derived by the use of the Lorentz contraction and the time dilation, does not hold for the current loop and consequently the relation (19) for the polarization

\mathbf{P} , which is induced by the movement of a permanent magnetization \mathbf{M}' , does not hold as well. The vector of the electric field outside a moving current loop is not caused by *the 3D polarization \mathbf{P}* , (19), but it could be determined by the LT, the same as (42) and (43), of *the vector of the electric field of the same but stationary current loop*.

7.2 The experiments for the detection of the second-order electric fields

The external second-order electric fields from a stationary, superconducting, current loop, i.e., coil, have not yet detected. However, in [28], an I^2 - dependent potential resulting from constant current in closed superconducting coils has been reported and the same happened in the first variation of experiments in [29], but not in the second one. It is worth mentioning that all these experiments are sensitive only to a monopole field and thus they cannot either support or disprove the theory presented in [27] and here, which predicts a dipole field; very small (second-order) external electric field. This happens because a direct contact with the superconducting coil is used in the measurements [28], [29] of the I^2 - dependent potential. In such measurements it can be only “seen” if some charge is created or destroyed. In order to directly measure the external electric fields it is necessary to use a non-contact method of measuring. Such a method is recently presented in [30]. The experiments [30] are “based essentially on the detection of a non-zero force between a circular steady current and a charge, both at rest in the Earth frame.” For the experimental setup see Fig. 1. in both papers in [30]. Observe that a Helmholtz coil γ that is used in the experiments is a *normal metal with finite resistivity*. The authors of [30] considered that they obtained a positive evidence for a non-zero force and that such results “show that *local Lorentz invariance could in fact be broken even in electromagnetic experiments ..*” However, their results have nothing to do with the breakdown of the local Lorentz invariance because, as mentioned above, the standard Maxwell theory predicts that there are always static magnetic fields and *a time independent electric field outside a normal conductor with a steady current*. Such an *external electric field* causes a non-zero force on a stationary charge q in the experiments in [30].

In order to “see” the existence of the external second-order electric fields the coil used in the experimental setup in [30] would need to be a superconducting coil. Hence, *we propose to experimentalists to make the similar measurements as in [30], but using a superconducting coil*. It would be an important test of the validity of the relations (53) and (54), i.e., of the relation (55), or the validity of the usual approaches which assume the Clausius hypothesis. Namely, it is often declared that the classical electromagnetic theory predicts a zero external electric field for a stationary superconducting coil. But, it is not true. Maxwell equations enable one to find fields in the case that the sources are known. Hence, Maxwell equations will give a zero external electric field only if one supposes that in a stationary superconductor with a steady current the local charge density (in the ions’ rest frame and in the standard basis) is everywhere zero. Inside the classical electromagnetic theory this statement that

the local charge density is everywhere zero is merely *a hypothesis*, the Clausius hypothesis. This means that the possible existence of the second-order external electric fields from steady currents in a stationary superconductor is not at all in contradiction with classical electromagnetism. However, as stated in [27] and mentioned above, in contrast to the usual approaches in which there is no either physical or mathematical justification for the Clausius hypothesis or for some other choice, e.g., $\rho'_- = -\rho_0$, in the 4D geometric approach Eqs. (53), (54) and (55) are the consequences of the use of correctly defined quantities in the 4D spacetime, i.e., the 4D geometric quantities. The fields E and B with components in the standard basis given by Eqs. (55) and (56), respectively, correctly transform under the LT (42) and (43), which is not the case with the 3-vectors \mathbf{E} and \mathbf{B} obtained in all previous approaches.

Recently, an interesting possibility to experimentally investigate these second-order external electric fields is proposed in [31]. As already mentioned above, the theory from [31] is not relativistically correct, but the proposed experiments with the cold ions could be the right way to detect such small electric fields. Moreover, the estimated size of that electric field could be even bigger than that one found in [31] if the experiments would be made using a superconducting coil that is wound using a large number of bifilar pairs.

Another possibility to study these second-order external electric fields by the use of a non-contact method of measuring could be similar to that one used to measure the Casimir force in [43], i.e., by the use of a torsion balance.

As stated in Sec. 3.3 in [27], the second-order electric fields could play an important role in many physical phenomena with steady currents, particularly in tokamaks and astrophysics, where high currents exist, and in superconductors, where the electric fields that are caused by surface charges are absent.

Regarding the role of the second-order electric fields in tokamaks, we have to mention that recently, [44] and references [1], [5] and [7] therein, the existence of the radial electric field $E_r(r)$ in quasi-neutral tokamak plasma is examined taking into account the Lorentz contraction of “an “electron ring” circumference in steady state tokamak plasma rotating in toroidal direction with current velocity $V_e(r) \dots$ ” The similar consideration with the Lorentz contraction of an “electron ring” was already reported in [45]. But, as we mentioned several times, such theories with the Lorentz contraction are not the relativistically correct theories. It would be very important for physics that the experimentalists find the best way for the direct and precise measurements of the second-order electric fields that are predicted in [27] and here by the relativistically correct approach with 4D geometric quantities.

8. The electric field outside a stationary permanent magnet

An interesting consequence of the above consideration refers to the existence of *the electric field outside a stationary permanent magnet*. It was mentioned at the end of [20]. Namely, as already stated, in the Ampèrian approach a permanent magnet is essentially an assembly of current loops. However, as discussed above, *the second-order external electric field exists not only for every*

moving current loop, but also for the same stationary current loop, which yields that the electric field would need to exist not only outside a moving permanent magnet but outside a stationary permanent magnet as well. This conclusion is also supported by the following argument. According to the LT (42) and (43), if there is an electric field outside the moving magnet it would necessary need to exist outside the same but stationary magnet. Thus, in the Ampèrian approach, there is a polarization vector P (remember that P is a 4D geometric quantity and not a 3-vector) for a stationary permanent magnet as an assembly of small current loops. As explained above, this happens because *every current loop behaves like an electric dipole at points far from that loop*. Then, that P induces an external electric field.

If we abandon the Ampèrian approach then, nevertheless, there is another explanation for the possibility that a stationary permanent magnet possesses an intrinsic polarization.

According to the well-known Uhlenbeck-Goudsmit hypothesis there is a connection between the 3-vectors of the magnetic moment \mathbf{m} of an electron and its spin \mathbf{S} , $\mathbf{m} = \gamma_S \mathbf{S}$. However, in the 4D spacetime, i.e., in the approach with 4D geometric quantities, the 3-vectors \mathbf{m} and \mathbf{S} are not properly defined quantities. In the 4D spacetime, as explained in [33], and in [46-48], the primary quantity (with independent physical reality) for the dipole moments is the dipole moment bivector D (four-tensor D^{ab} in [33], [46-49]) of a fundamental particle. It is decomposed into the EDM vector p , the MDM vector m and the unit time-like vector u/c , where u is the velocity vector of the particle. Then, p and m are derived from D and the velocity vector of the particle u according to the equations

$$\begin{aligned} D &= (1/c)[p \wedge u + (mI) \cdot u/c], \\ p &= D \cdot u/c, \quad m = I(D \wedge u), \end{aligned} \quad (57)$$

Eq. (2) in [33] (but in the tensor notation). It holds that $p \cdot u = m \cdot u = 0$; only three components of p and three components of m are independent since \mathcal{M} is antisymmetric. In the particle's rest frame (the K' frame) and the standard basis $\{\gamma'_\mu\}$, $u = c\gamma'_0$, and using (57), it follows that $p'^0 = m'^0 = 0$, $p'^i = D'^{i0}$, $m'^i = (c/2)\epsilon^{0ijk}D'_{jk}$. Therefore p and m can be called the “time-space” part and the “space-space” part, respectively, of the dipole moment bivector D . But, these parts are written with quotation marks because in all other relatively moving frames and in all other bases the above identifications of the components of p and m with the components of D do not hold.

Similarly, it is shown in [33] (earlier, in [48] and Ref. [3] in [48]) that the primary quantity *with definite physical reality* for the *intrinsic* angular momenta is the spin bivector \mathcal{S} (four-tensor S^{ab} in [33], [48]), which is decomposed into the usual “space-space” intrinsic angular momentum S , the “time-space” intrinsic angular momentum Z and the unit time-like vector u/c , where u is the velocity vector of the particle

$$\begin{aligned} \mathcal{S} &= (1/c)[Z \wedge u + (SI) \cdot u], \\ Z &= \mathcal{S} \cdot u/c, \quad S = I(\mathcal{S} \wedge u), \end{aligned} \quad (58)$$

Eq. (8) in [33]. It holds that $Z \cdot u = S \cdot u = 0$; only three components of Z and three components of S are independent since \mathcal{S} is antisymmetric. S and Z depend not only on \mathcal{S} but on u as well. Only in the particle's rest frame, the K' frame, and the $\{\gamma'_\mu\}$ basis, $u = c\gamma'_0$ and $S'^0 = Z'^0 = 0$, $S'^i = (1/2c)\varepsilon^{0ijk}S'_{jk}$, $Z'^i = S'^i$. According to Eq. (58), a new “time-space” spin Z is introduced and it is a physical quantity in the same measure as it is the usual “space-space” spin S .

Then, in [33], it is suggested that instead of the connection between the 3-vectors \mathbf{m} and \mathbf{S} we need to have the connection between the dipole moment bivector D and the spin bivector \mathcal{S} , which is formulated in the form of the generalized Uhlenbeck-Goudsmit hypothesis

$$D = g_S \mathcal{S}, \quad (59)$$

Eq. (9) in [33]. Hence, inserting the decompositions of D (57) and \mathcal{S} (58) into Eq. (59) we find the connections between the dipole moments m and p and the corresponding intrinsic angular momenta S and Z , respectively,

$$m = cg_S S, \quad p = g_S Z, \quad (60)$$

Eq. (10) in [33]. In the particle's rest frame, the K' frame, and the $\{\gamma'_\mu\}$ basis, $u = c\gamma'_0$ and $p'^0 = m'^0 = 0$, $p'^i = g_S Z'^i$, $m'^i = cg_S S'^i$. Comparing this last relation with $\mathbf{m} = \gamma_S \mathbf{S}$, we see that $g_S = \gamma_S/c$. In contrast to all previous approaches with the 3-vectors in which both the MDM \mathbf{m}' and the EDM \mathbf{p}' of an elementary particle are determined by the usual spin \mathbf{S}' , we find that the intrinsic MDM vector m of an elementary particle is determined by the “space-space” spin vector S , whereas the intrinsic EDM vector p is determined by the “time-space” spin vector Z . The relations (59) and (60) show that any fundamental particle has not only the intrinsic MDM m , but also the intrinsic EDM p whose magnitude is $(1/c)$ of that for m . The EDM p (see (60)) emerges from the connection with the intrinsic angular momentum Z , i.e. from (59) and (57), (58) in the same way as the MDM m emerges from the connection with the intrinsic angular momentum S . The EDM p is an intrinsic property of a fundamental particle in completely the same way as it is the intrinsic MDM m . As stated in [33]: “The EDM obtained in this way is of quite different physical nature than in the elementary particle theories, e.g., in the standard model and in SUSY. There, an EDM is obtained by a dynamic calculation and it stems from an asymmetry in the charge distribution inside a fundamental particle, which is thought of as a charged cloud.”

These fundamental results for the generalized Uhlenbeck-Goudsmit hypothesis (59), i.e., for the new spin Z and the associated EDM p of a fundamental particle (60) are used in [33] and [49] for the discussion of the shortcomings in experimental searches for a permanent EDM of particles. In all these searches the AT of \mathbf{E} and \mathbf{B} (10) and the AT for the 3D EDM \mathbf{p} and MDM \mathbf{m} (18) are always considered to be relativistically correct, i.e., that they are the LT. Furthermore, it is supposed in the elementary particle theories, that not only the 3D MDM \mathbf{m} is proportional to the usual 3D spin \mathbf{S} ($\mathbf{m} = m(\mathbf{S}/S)$), the

Uhlenbeck-Goudsmit hypothesis), but the 3D EDM \mathbf{d} as well. The results from [33] and [49] explicitly show that from the ISR viewpoint this basic assumption is not true and that it has to be replaced by the relation (60) which deal with 4D geometric quantities that are correctly defined in the 4D spacetime.

If the spins and the dipole moments are quantized, i.e., if they become operators, then, in [48], the commutation relations for the components in the standard basis of the intrinsic angular momentums S and Z are given by Eq. (4) in [48]

$$\begin{aligned} [S^\mu, S^\nu] &= (i\hbar/c)\varepsilon^{\mu\nu\alpha\beta}S_\alpha u_\beta, \quad [Z^\mu, Z^\nu] = (-i\hbar/c)\varepsilon^{\mu\nu\alpha\beta}S_\alpha u_\beta, \\ [S^\mu, Z^\nu] &= (i\hbar/c)\varepsilon^{\mu\nu\alpha\beta}Z_\alpha u_\beta. \end{aligned} \quad (61)$$

Taking into account the relation (60) the commutation relations for the components of m and p , m^μ and p^μ respectively, are expressed in terms of those for S^μ and Z^μ , Eq. (61), and they are given by Eq. (5) in [48]

$$[m^\mu, m^\nu] = c^2 g_S^2 [S^\mu, S^\nu], \quad [p^\mu, p^\nu] = g_S^2 [Z^\mu, Z^\nu], \quad [m^\mu, p^\nu] = c g_S^2 [S^\mu, Z^\nu]. \quad (62)$$

As explained above, an electron possesses both intrinsic angular momentums, spins, S and Z and, according to (60), the associated dipole moments m and p , respectively. All these quantities, S and Z , m and p are vectors, properly defined geometric quantities in the 4D spacetime. They transform according to the LT, the same as (42) and (43). This means that *under the LT vector m transforms again to the magnetic dipole moment* and, contrary to the LPET of the 3-vectors \mathbf{m} and \mathbf{p} (the same as (16)) *there is no mixing with p* . The same holds for vectors S and Z .

At this point we present a simple discussion, only qualitative arguments, about the existence of an electric field outside a stationary permanent magnet and about the experimental detection of that field. It can be concluded that in the same way as the MDMs determine the magnetization M of a stationary permanent magnet the EDMs determine its polarization P , which induces an electric field outside a permanent magnet (moving or *stationary*). Note that according to the LT, the same as (42) and (43), if there is a polarization vector P in one inertial frame of reference it will exist in all other relatively moving inertial frames of reference. As $g_S = \gamma_S/c$, the EDM p is in magnitude $1/c$ of the MDM m . Consequently, the external electric field is much smaller than the magnetic field outside a stationary permanent magnet. It can be one of the reasons why such electric fields are not yet experimentally detected. Another, equally important reason, is that the experimentalists never looked for such fields because it was no theory that predicts them.

In order to directly measure the external electric fields from a stationary permanent magnet it is necessary to use a non-contact method of measuring. A possible experimental setup for the detection of that field could be again with the cold ions as in [31]. Another way of measuring such electric fields from a

stationary permanent magnet would be some modification of torsion balance setup that is used in [43] for the detection of the Casimir force.

9. “Charge-magnet paradox” and its resolution with the 4D torques

9.1 Criticism of the “resolutions” [34-36] of Mansuripur’s paradox

A nice example which clearly illustrates the essential difference between the 4D geometric approach, i.e., the ISR, with relativistically correct LT and Einstein’s formulation of SR with the AT of the 3-vectors refers to the recently reported [34] “charge-magnet paradox,” to the highlight of it [35] and different “resolutions” [36]. In [34], it is argued that in the presence of the magnetization \mathbf{M} and the electric polarization \mathbf{P} the usual expression for the Lorentz force with the 3-vectors fails to accord with the principle of relativity, because it leads to an apparent paradox involving a MDM \mathbf{m} in the presence of an electric field \mathbf{E} ; in a static electric field a MDM \mathbf{m} is subject to a torque \mathbf{T} in some frames and not in others. (Henceforward we shall denote the usual 3D torque by \mathbf{T} .) Mansuripur [34] argues that the conventional Lorentz force (density), Eq. (5) in [34], should be replaced by the Einstein-Laub law, Eq. (6) in [34], which predicts no torque \mathbf{T} in all frames. The paradox is reminiscent of the Trouton-Noble paradox [50] and of the Jackson paradox [51]. For the resolution of these paradoxes with the 4D geometric quantities, i.e., in the ISR, see [12], [13] and [14], respectively. The paper [34] and the paper [35] that highlighted it and almost all papers in [36] that objected it deal with the 3-vectors and their LPET (10) for \mathbf{E} and \mathbf{B} , (16), or (17) for \mathbf{P} and \mathbf{M} , and the same for EDM \mathbf{p} and MDM \mathbf{m} , i.e., (18) or (19), as that these transformations are the relativistically correct LT. For example, in [34], it is stated: “Lorentz transformation to the xyz frame then yields a pair of point dipoles, ...,” see also Eqs. (9 - 10b) and Eqs. (11 - 12b). Furthermore, Eq. (18), or (19), is taken as the main point in the majority of the expositions in [34-36]. But, these transformations are the AT and not the LT. This means that the requirement like (49) does not hold either for the conventional Lorentz force or the Einstein-Laub expression and also for the corresponding 3D torques. All these quantities are not well-defined quantities in the 4D spacetime. Similarly happens for all papers from [34-36]. Hence, as will be shown in this section, from the ISR viewpoint, the resolutions of the “charge-magnet paradox” that are proposed in [34-36] are not the relativistically correct resolutions.

In [34], in the discussion of another example, not only that LPET (10) and (16) are considered to be the LT, but the equation $\mathbf{F} = d\mathbf{p}/dt$ is considered to be “the relativistic version of Newton’s law.” There, it is stated “.. its relativistic momentum \mathbf{p} increases with time, not because of a change of velocity but because of a change of mass.” It is shown in the preceding sections, i.e., in [6-11], that the LPET ARE NOT the LT, and as shown in detail in, e.g., [14], [10], any 3D quantity cannot correctly transform under the LT; it is not the same quantity for relatively moving observers in the 4D spacetime. *The equation $\mathbf{F} = d\mathbf{p}/dt$ with $\mathbf{p} = m\gamma_u \mathbf{u}$ is not the relativistic equation of motion* since, contrary to the common assertions, e.g., [2], [22], [26], it does not retain

the same form in two relatively moving inertial frames S and S' , i.e., it is not covariant under the LT, see, e.g., the discussions in Sec. 3 in the first paper in [14] and particularly in [10]. In that equation the primed 3D quantities are not obtained by the LT from the unprimed ones, but they are obtained in terms of the AT for the 3D force \mathbf{F} , e.g., Eqs. (12.66) and (12.67) in [26], and the 3D momentum \mathbf{p} , i.e., the 3D velocity \mathbf{u} , Eq. (11.31) in [2]. The requirement like (49) does not hold for any of the mentioned 3D quantities. Instead of the equation with the 3D quantities one has to use the equation of motion with 4D geometric quantities, Eq. (10) in [14], $K = dp/d\tau$, $p = mu$, where p is the proper momentum vector and τ is the proper time, a Lorentz scalar. For the definition of the Lorentz force K_L

$$K_L = (q/c)F \cdot u = (q/c) [(1/c)E \wedge v + (IB) \cdot v] \cdot u, \quad (63)$$

see, e.g., the first paper in [14]. The quantities K (K_L), p , u transform in the same way, like any other vector, i.e., according to the LT, the same as (42), (43) and not according to the awkward, AT of the 3-force \mathbf{F} and the 3-momentum \mathbf{p} , i.e., the 3-velocity \mathbf{u} . Moreover, only the rest mass is well defined quantity in the 4D spacetime and thus there is not “a change of mass.”

In [35], it is tried to explain why the moving magnet appears to be electrically polarized. The explanation is completely the same as it is the derivation of the relations (18) and (19) in [23] and [22] and in Sec. 3.2 here, i.e., it explicitly uses *the Lorentz contraction*. But, as can be seen, e.g., from Appendix here, the Lorentz contraction is an AT, i.e., it is ill-defined in the 4D spacetime. Similar consideration with the use of *the Lorentz contraction* is presented in Unnikrishnan’s paper in [36].

In some papers from [36], e.g., Griffiths and Hnizdo (GH), McDonald, Saldanha, the model of the “hidden” momentum and the “hidden” angular momentum is invoked in order to show that, contrary to the claims in [34], the usual Lorentz force law with the 3-vectors is consistent with SR. Their approach is also with the 3-vectors and the LPET, i.e., the AT, and, particularly, in all three papers it is taken that the moving magnetic dipole, i.e., an Ampèrian loop, acquires an electric dipole moment given by Eq. (18). Let us explain in some detail the approach from GH [36]. They find that if the magnetic dipole is considered to be an electric current loop then the net torque \mathbf{T} (their \mathbf{N}) on the dipole, Eq. (4) in GH [36], is non-zero in the lab frame, i.e., the principle of relativity is violated. Observe that \mathbf{T} from their Eq. (4) is obtained from the term $\mathbf{p} \times \mathbf{E}$, where \mathbf{p} is the induced electric dipole moment, their Eq. (3), i.e., our Eq. (18), which means that *the violation of the principle of relativity is a direct consequence of the AT for the 3-vectors \mathbf{p} and \mathbf{m}* . To resolve the problem they introduce two sorts of the angular momentums “.. \mathbf{L}_o is the “overt” angular momentum (associated with actual rotation of the object), and \mathbf{L}_h is the “hidden” angular momentum (so called because it is *not* associated with any overt rotation of the object),” $\mathbf{L} = \mathbf{L}_o + \mathbf{L}_h$, where \mathbf{L}_h is given by their Eq. (10). Furthermore, they define the torque \mathbf{T} by their Eq. (12), $\mathbf{T} = d\mathbf{L}/dt = d\mathbf{L}_o/dt + d\mathbf{L}_h/dt$ and state “*Physically $d\mathbf{L}_h/dt$ is not a torque,*

(my emphasis) but (the rate of change of) a piece of the angular momentum ..” In order to get the agreement with the principle of relativity they argue that it is not \mathbf{T} , which has to be zero in both frames, but it is the “effective” torque \mathbf{T}_{eff} given by their Eq. (13). Thus, they assert: “.. it is the “overt” torque (given by Eq. (13), my remark) that must vanish to resolve the paradox, since the dipole is not rotating ..”

Several objections to their treatment are at place already here. (i) First of all, as already mentioned, their approach deals with the 3-vectors and their AT and not with the 4D quantities and their LT. The requirement like (49) is not fulfilled for any 3D quantity. (ii) In such an approach with 3-vectors the introduction of the “hidden” quantities is very artificial and it is not justified either mathematically or physically. (iii) The derivatives in Eqs. (11-13) are over the *coordinate time* and not over the proper time, which means that neither of them can properly transform under the LT. (iv) Their assertion about $d\mathbf{L}_h/dt$ is meaningless. It cannot be that in an equation (their Eq. (13)) with physical quantities \mathbf{T} is a torque and $d\mathbf{L}_o/dt$ is also a torque, but $d\mathbf{L}_h/dt$ is *not* a torque. Obviously, in the usual formulation with the 3-vectors there is no proper physical interpretation for $d\mathbf{L}_h/dt$, i.e., for different “hidden” quantities.

It can be seen from the resolution of Jackson’s paradox [14], the Trouton-Noble paradox [12, 13] and from Sec. 9.2 here that if the physical reality is attributed to the 4D geometric quantities which properly transform under the LT and not, as usual, to the 3D quantities which transform according to the AT, then all quantities are mathematically and physically correctly defined and there is no “hidden” quantity, i.e., the quantity for which there is not some proper physical interpretation in the 4D spacetime. For example, the angular momentum is not the 3-vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and the torque is not $\mathbf{T} = \mathbf{r} \times \mathbf{F}$ with $\mathbf{T} = d\mathbf{L}/dt$, but they are the abstract 4D quantities, here the bivectors,

$$J = x \wedge p, \quad N = x \wedge K; \quad N = dJ/d\tau, \quad (64)$$

where x is the position vector, p is the proper momentum vector, $p = mu$, u is the proper velocity vector $u = dx/d\tau$ of a charge q (it is defined to be the tangent to its world line), τ is the proper time and K is the force vector. If J and N are written as CBGQs in the $\{\gamma_\mu\}$ basis they are given by Eq. (12) in the first paper in [14],

$$\begin{aligned} J &= (1/2)J^{\mu\nu}\gamma_\mu \wedge \gamma_\nu, \quad J^{\mu\nu} = m(x^\mu u^\nu - x^\nu u^\mu), \\ N &= (1/2)N^{\mu\nu}\gamma_\mu \wedge \gamma_\nu, \quad N^{\mu\nu} = x^\mu K^\nu - x^\nu K^\mu. \end{aligned} \quad (65)$$

The components $J^{\mu\nu}$ from (65) are identical to the usual covariant angular momentum four-tensor and similarly for $N^{\mu\nu}$.

Cross [36], in the Comment on Mansuripur’s paper, states: “The torque density is not a vector, but the antisymmetric tensor ..” and also “.. the angular momentum, which is the second rank tensor ..” So, he correctly notices that the angular momentum and the torque density are not the 3D vectors. However, his quantities $J^{\alpha\beta}$ and $T^{\alpha\beta}$ also are not tensors, but components implicitly taken in the standard basis as in the usual covariant approaches, e.g., from [2],

[22], [26]. They correspond to $J^{\mu\nu}$ and $N^{\mu\nu}$ from (65). Tensors are abstract, geometric quantities, e.g., in the abstract index notation M^{ab} , Eq. (4) in [33], which corresponds to the bivector J in (64) in the geometric algebra formalism. If tensors are represented in some basis then they become the CBGQs that contain components *and the basis vectors* as in Eq. (65). As already mentioned several times, if one does not use Einstein's synchronization but, e.g., the "r" synchronization then it is not possible to make the identification of the components of the 3-vectors \mathbf{E} and \mathbf{B} with the components $F^{\alpha\beta}$ of the electromagnetic field F , or the components of the 3-vectors \mathbf{R} and \mathbf{T} (in Cross's notation) with the components $T^{\alpha\beta}$ of the torque tensor, etc. This means that the physical laws have to be written with the abstract quantities and then to represent these quantities in some basis as the CBGQs. Furthermore, he states, "For short we write $M = (\mathbf{P}, -\mathbf{M})$ in terms of the defining 3-vectors." Similarly, he writes " $F = (-\mathbf{E}, -\mathbf{B})$ is the Faraday tensor .." and also " $m = (\mathbf{p}, -\mathbf{m})$ is the tensor of moments." Hence, in his covariant approach with "tensors," in the same way as in all traditional approaches, it is considered that the 3-vectors are the primary quantities with a definite physical reality, whereas $M^{\alpha\beta}$, $F^{\alpha\beta}$, $m^{\alpha\beta}$, etc. are only auxiliary mathematical quantities that are defined by the components of the 3-vectors.

The situation is completely different in the ISR in which the primary quantities are the abstract quantities, here the bivectors, thus not only components, \mathcal{M} , F , D (our notation), etc. As seen from Eqs. (25) and (26), Eqs. (21) and (22), Eq. (57), etc. P and M are derived from \mathcal{M} and they depend not only on \mathcal{M} but on u as well, and similarly for the bivector F and vectors E , B and v and also for D and p , m and u .

Note that Cross [36] also deals with the "hidden" quantities relating them to the time-space components, e.g., N^{0i} (T^{0i} in Cross's notation) of the torque tensor. He argues: ".. that the torque predicted in the moving frame is correct and necessary to balance the "hidden" angular momentum of the moving dipole rather than causing a precession of the spin." This means that under the torque he does not understand the whole torque tensor, but only the 3D torque \mathbf{T} that is related with the rotation in the 3D space. Indeed, for the time-space components, i.e., for the components of his \mathbf{R} it is stated: "These "torque" components are connected with the motion of the center of energy .."

In the 4D spacetime the center of energy is not well-defined quantity. As seen from [12-14] and as will be seen below such understanding is in a sharp contrast to ISR in which *only* the 4D geometric quantities are well-defined in the 4D spacetime, e.g., the torque bivector N and the torque vectors N_s and N_t , see Eq. (74).

Observe also that Cross [36], in contrast to all others from [34-36], showed that the offending torque $T'^x = \gamma v m E$ (in his notation) that is obtained in a non-covariant analysis, i.e., with the 3-vectors and their AT for EDM \mathbf{p} and MDM \mathbf{m} , Eq. (18), is determined as the Lorentz transformed time-space component that exists even in the rest frame. In his notation that nonvanishing component is $R^y \equiv T^{ty} = -mE$. Then, he states: "Under a Lorentz boost to the moving frame the space-space and time-space components mix." Hence, his

3D torque $T'^x = \Lambda_t^z T^{yt}$ is considered to be obtained by the LT. However, the comparison of his derivation of T'^x with the derivation of the AT (10), i.e., (9), in Sec. 3.1 reveals that both derivations are exactly the same. Indeed, Cross [36] simply assumes that the components of the 3-vectors \mathbf{R} and \mathbf{T} are identified with the components $T^{\alpha\beta}$ of the torque tensor *in both frames* (this corresponds to Eqs. (3) and (7)) and accordingly they transform like the components $N^{\alpha\beta}$ ($T^{\alpha\beta}$ in Cross's notation) transform. For comparison, the reader can look at the text in Sec. 3.1 between Eq. (9) and Eq. (10). Thus, contrary to the assertion in Cross [36] the transformations for T'^x are not the LT, but they are exactly the same as the LPET, i.e., the AT (10). The explicit AT of the components of the 3-vectors \mathbf{T} and \mathbf{R} are given, e.g., by Eq. (19) in [13], which is repeated here

$$\begin{aligned} T_1 &= T'_1, \quad T_2 = \gamma(T'_2 - \beta R'_3), \quad T_3 = \gamma(T'_3 + \beta R'_2), \\ R_1 &= R'_1, \quad R_2 = \gamma(R'_2 + \beta T'_3), \quad R_3 = \gamma(R'_3 - \beta T'_2). \end{aligned} \quad (66)$$

They are written for the motion along the x^1 axis and not along the x^3 axis as in Cross [36] and also the components $T_{t,i}$ from Eq. (19) in [13] are replaced by R_i . With these changes, the component in (66) that corresponds to T'^x in Cross [36] is $T_2 = -\gamma\beta R'_3$. It is visible from (66) that the transformations for R_i are the same as the AT for E_i in Eq. (9) and similarly is for T_i and B_i . The essential point is that, e.g., *the components T_i of the torque 3-vector \mathbf{T} in the moving frame are expressed by the mixture of the components of the 3D vector \mathbf{T}' and of another 3D vector \mathbf{R}' from the rest frame*. This is the reason that the components of the usual 3D torque \mathbf{T} will not vanish in the S frame even if they vanish in the S' frame, i.e., that there is the “charge-magnet paradox” in all usual approaches to special relativity that deal with the 3-vectors or with components implicitly taken in the standard basis. The same discussion as in Sec. 6 after Eq. (49) can be repeated for the 3-vectors \mathbf{T} and \mathbf{R} . Again it can be stated that *as far as relativity is concerned the quantities, e.g., \mathbf{T} and \mathbf{T}' are not related to one another* and that their identification is simply a *mistaken identity*.

However, as can be seen from Eq. (82) below, the situation is completely different for the 4D torque N_s , and the same for N_t . The physical 4D torque, the basis-free, abstract, N_s , simply has different representations, 4D CBGQs, in different bases.

Very similar procedure as in GH [36] is commonly used for the resolution of the Trouton-Noble paradox, see Refs. [3-7] in [13]. In the Trouton-Noble paradox, in the similar way as in [34], there is a 3D torque and so a time rate of change of 3D angular momentum in one inertial frame, but not in another relatively moving inertial frame. For the usual resolution of that paradox it is taken that *there is another 3D torque*, which is equal in magnitude but of opposite direction, giving that the total 3D torque is zero in order to have agreement with the principle of relativity. Different explanations have been offered for the existence of that additional 3D torque, e.g., the nonelectromagnetic forces with

their additional torque, Refs. [3-6] in [13], or the field angular momentum and its rate of change, i.e., its additional torque, Ref. [7] in [13].

The resolution of Mansuripur's paradox that is presented by McDonald [36] is almost the same as in GH [36] with the difference that McDonald introduces the field angular momentum and states: "... the "paradoxical" nonzero torque is needed to change the "hidden" mechanical angular momentum of the system, such that this remains equal and opposite to the field angular momentum, .. ." The above objections (i) - (iii) hold in the same measure for McDonald's approach with an additional objection that the usual expressions for the electromagnetic momentum that are given by his Eq. (1) are all with the 3D quantities, which means that they also are not well-defined in the 4D spacetime; they do not properly transform under the LT.

Kholmetskii, Missevitch and Yarman [36] exclusively deal with 3D quantities and introduce the contribution of the hidden momentum as well.

Milton and Meille [36] also deal with the 3-vectors and the AT (10) for \mathbf{E} and \mathbf{B} . Particularly, for a point dipole they argue that: "... the torque is not zero, but is balanced by the rate of change of the angular momentum of the electromagnetic field, so there is no mechanical torque on the dipole." Their 3D quantities also do not properly transform under the LT.

Brachet and Tirapegui [36] combine the usual covariant approach with components implicitly taken in the standard basis with the formulation in terms of the 3-vectors writing, e.g., the energy density u and the Poynting 3-vector \mathbf{S} in terms of the 3-vectors \mathbf{E} and \mathbf{B} and the integrations in $\mathcal{P}_{\text{field}}$ and $\mathcal{L}_{\text{field}}$ are over the 3D space. In the lab frame they have the torque, a 3D quantity, but they argue that "it is required to account for the motion at uniform speed of the missing momentum $\mathcal{P}_{\text{particle}}$." Thus, they also deal with quantities which do not correctly transform under the LT.

Boyer [36], as almost all others, defines the relativistic conservation laws, the Lorentz transformations of forces, the Lorentz transformations of energy and momentum, the angular momentum and torques, etc., all in terms of 3D quantities, e.g., the 3-vectors \mathbf{E} , \mathbf{B} , \mathbf{r} , \mathbf{p} , \mathbf{F} , \mathbf{L} , etc., considering that their AT are the LT. It is interesting that only Boyer in his Ref. 6 mentioned my Comment (arXiv: physics/0505013) to Jacson's paper [51] as that it was published. How can it be that *in the 4D spacetime* the 3D quantities are considered to be well-defined, whereas the 4D geometric quantities are considered to be ill-defined?

For the electromagnetic momentum that correctly transforms under the LT see Secs. 4, 5 - 5.3 in [52] (only components) and for the more general expressions with the 4D geometric quantities see Sec. 2.6 in [12]. There, in [12], an axiomatic geometric formulation of electromagnetism with the Faraday bivector field F as the primary quantity for the whole electromagnetism is presented and in Sec. 2.6 the observer independent expressions with the abstract quantities, Eqs. (37) - (43), for the stress-energy vector $T(n)$, the energy density U , the Poynting vector S and the momentum density g , the angular momentum density M and the Lorentz force K_L are directly derived from the field equation for F , Eq. (4) in [12] (the notation is that one from [12]). These quantities are also written as CBGQs in the standard basis, Eqs. (44) - (47). Furthermore, the

local conservation laws are directly derived from that field equation for F and presented in Sec. 2.7 in [12], see Eqs. (48) - (51).

In [36], only in Vanzella's paper the CBGQs in the standard basis are used, but again the formulation with such quantities is combined with the formulation in terms of the 3-vectors. He "mimic the magnet by a neutral ring conducting an electric stationary current I ." Although he does not explicitly use the AT of the 3D quantities, the assumption that $j^\mu u_\mu = 0$, "reflecting the neutrality of the ring according to \mathcal{O} ," where \mathcal{O} is an observer at rest in the frame of the ring, is equivalent to the Clausius hypothesis. According to the discussion in Sec. 7.1 that hypothesis is not well justified in the 4D spacetime. That assumption leads to the result that for a relatively moving observer the ring is polarized and there is an induced electric dipole moment 3-vector and a 3D torque $\tau = \mathbf{d} \times \mathbf{E}$ (his notation) on the ring. That torque is the same as that the AT (18) are explicitly used. Furthermore, he declares: "the torque exerted by the electric field is not used to rotate the ring but rather to move its asymmetric distribution induced by the very same the electric field. No paradox here." In principle, such a conclusion is very similar to that one as in GH [36] or Cross [36]. Griffiths and Hnizdo [36] and also Cross [36] consider that there is no paradox because the offending torque (3-vector) is not "associated with actual rotation of the object." In the 4D spacetime their resolution of the paradox is not correct, because it is based on the use of the 3D quantities and their AT. As shown below the relativistically correct torques are the bivector N and the vectors N_s and N_t .

Griffiths and Hnizdo [36] assert: "This "paradox" was resolved many years ago by Victor Namias [4]." But, it is not true. Namias, as almost all others, exclusively dealt with the 3D quantities and their AT. Similarly, Vanzella [36] declares: "... there is a very similar and famous "paradox" (the Trouton-Noble "paradox" [2]) which was presented and resolved more than a hundred years ago [3] (see also Refs. [4, 5])." As can be seen from [12-14] that Vanzella's statement is not correct. The "resolutions" presented in Vanzella's references [3-5] dealt with different AT, the Lorentz contraction, the dilation of time and the LPET (10) and (16) considering them as that they are the relativistically correct LT.

In addition, it is worth mentioning that *all treatments from [34-36] are meaningless* if some other basis and not the standard basis is taken into account, e.g., *if only the Einstein synchronization is replaced by the "radio" synchronization*. This conclusion simply follows already from Eqs. (4) and (5).

9.2 The resolution of Mansuripur's paradox with the 4D torques

The geometric approach to the "charge-magnet paradox" that is presented here significantly differs from all formulations in [34-36]. First of all, in contrast to [34-36], this approach deals from the outset with 4D geometric quantities, their LT and equations with them. It is already shown in the similar treatments of the Trouton-Noble paradox [12, 13] and Jackson's paradox [14] that in the approach with 4D geometric quantities the principle of relativity is naturally satisfied and there is no paradox. Every 4D geometric quantity is the same

quantity for all relatively moving inertial observers and for all bases chosen by them. The equation like (49) holds for any such quantity. Furthermore, the most important difference is that the torque bivector is different from zero even in the common rest frame of the considered charge and magnet from [34]. The reason for that important difference is that according to the 4D geometric approach, i.e., the ISR, a stationary permanent magnet possesses not only the magnetization vector M but the polarization vector P as well. Observe that, according to the preceding discussion, instead of to speak about a permanent magnet one can equivalently consider a current loop with a steady current.

Let us consider the system from [34], but, without loss of generality, the electric charge will be substituted by a uniform electric field. The common rest frame of the source of the electric field (a point charge q in [34]) and of the permanent magnet will be denoted as S' , whereas the lab frame, in which the S' frame moves with uniform velocity $V = V\gamma_1$ along the common x_1, x'_1 axes, will be denoted as S . (Here, such a choice is taken for easier comparison with [13] and [14].) The treatment of the interaction between a static electric field and a permanent magnet that is presented here is very similar to the formulations given in [12-14]. For this example, the Lorentz force density k_L is given by Eq. (35) in which it is taken that $j^{(C)} = 0$, i.e., $k_L = (1/c)F \cdot j^{(M)}$, where $j^{(M)}$ is given by Eq. (31) and $F = (1/c)E \wedge v$, what is Eq. (21) with $B = 0$. Remember that *the vector B is zero in all relatively moving inertial frames of reference* and therefore there is no reason for the appearance of the paradox. This is an essential difference between the approach with the 4D geometric quantities and their LT and the usual approach with the 3D quantities and their AT. Compare with Eqs. (11 - 12b) in [34]. Hence, k_L from (35) becomes

$$k_L = (1/c^2)(E \wedge v) \cdot [-(\partial \cdot P)u + (u \cdot \partial)P + (1/c)[u \wedge (\partial \wedge M)]I]. \quad (67)$$

Observe that the expression for k_L contains two velocity vectors, v - the velocity vector of the observers who measure E and B fields and u - the velocity vector of the permanent magnet, i.e., of the electric current loop. The Lorentz force density k_L (67) can be written as a CBGQ in the standard basis and its form can be easily inferred from the expression for the torque density n (69) that is given below. Here, it will be presented the expression for k_L in the S' frame, i.e., for the case that $u = v = c\gamma'_0$ and accordingly that $E'^0 = P'^0 = M'^0 = 0$. Hence, k_L as a CBGQ in the $\{\gamma'_\mu\}$ basis is

$$k_L = (-E'^k \partial'_0 P'_k + (1/c)\varepsilon^{0jkl} E'_j \partial'_k M'_l) \gamma'_0 - E'^i (\partial'_k P'^k) \gamma'_i. \quad (68)$$

In the usual approaches with the 3-vectors and their AT, e.g., in [34] and in GH [36], the Lorentz 3-force density is zero in the S' frame; there is not the γ'_0 term and there is not the polarization 3-vector \mathbf{P} . The components $k_L'^\mu$ in (68) correspond to the time and spatial components of f^α from [36] (Cross), i.e., to $f^0 = (1/c)\mathbf{E} \cdot (\partial \mathbf{P} / \partial t + \nabla \times \mathbf{M})$ and $f^i = -E^i (\nabla \cdot \mathbf{P})$. But, as already stated, it is not correct to write the components of the properly defined vector in the 4D spacetime in terms of the 3-vectors. The components $k_L'^\mu$ are multiplied by

the unit basis vectors γ'_μ , whereas the 3-vector, e.g., \mathbf{E} is constructed from the components $E_{x,y,z}$ and the unit 3-vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

The torque density n is $n = x \wedge k_L$, where k_L is given by Eq. (67). If n is written as a CBGQ in the standard basis it becomes

$$n = (1/c^2)x^\lambda \{ (\partial_\mu P^\mu) [(E^\nu u_\nu)v^\rho - (v^\nu u_\nu)E^\rho] - (u^\mu \partial_\mu)P^\nu [E_\nu v^\rho - v_\nu E^\rho] + (1/c)\varepsilon^{\mu\nu\alpha\beta}u_\mu(\partial_\alpha M_\beta)[E_\nu v^\rho - v_\nu E^\rho] \} (\gamma_\lambda \wedge \gamma_\rho). \quad (69)$$

n from (69) will be determined in the S' frame in the same way as k_L is determined in (68), i.e., with $u = v = c\gamma'_0$, and $E'^0 = P'^0 = M'^0 = 0$,

$$n = [-(\partial'_k P'^k)x'^0 E'^i + (\partial'_0 P'^k)E'_k x'^i - (1/c)\varepsilon^{0jkl}E'_j(\partial'_k M'_l)x'^i] (\gamma'_0 \wedge \gamma'_i) - (\partial'_k P'^k)x'^j E'^i (\gamma'_j \wedge \gamma'_i) \quad (70)$$

It is worth noting that in the approaches with the 3-vectors, as for k_L in (68), the torque density n is zero in the S' frame.

In S' , the integrated torque N as a CBGQ is given as

$$N = -(1/c)E'^1 m'^2 (\gamma'_0 \wedge \gamma'_3) - E'^1 p'^3 (\gamma'_1 \wedge \gamma'_3), \quad (71)$$

where m is the magnetic dipole moment vector and p is the electric dipole moment vector. All quantities in (71) are measured in the common rest frame S' . They are all properly defined in the 4D spacetime and they properly transform under the LT. Furthermore, in the considered case, the electric field vector $E = E'^1 \gamma'_1$, the MDM $m = m'^2 \gamma'_2$ and the EDM $p = p'^3 \gamma'_3$. Using (71), the LT of the components $N'^{\mu\nu}$ and the inverse LT of the basis $\gamma'_\mu \wedge \gamma'_\nu$ from S' to S it can be shown that in S , the lab frame, the bivector N is given as

$$N = (-E^1 m^2/c + \beta E^1 p^3)(\gamma_0 \wedge \gamma_3) + (\beta E^1 m^2/c - E^1 p^3)(\gamma_1 \wedge \gamma_3), \quad (72)$$

and it holds that the whole 4D torque N is unchanged

$$N = (1/2)N'^{\mu\nu} \gamma'_\mu \wedge \gamma'_\nu = (1/2)N^{\mu\nu} \gamma_\mu \wedge \gamma_\nu, \quad (73)$$

where the LT of the electric field vector $E'^1 = \gamma(E^1 - \beta E^0)$, $E'^0 = \gamma(E^0 - \beta E^1) = 0$ are used to derive that $E'^1 = (1/\gamma)E^1$.

All quantities in (72) are measured in S in which the permanent magnet and the source of the electric field move with uniform velocity $V = V\gamma_1$ along the common x_1, x'_1 axes. The relations (71), (72) and (73) show that in the approach with the 4D torque N the principle of relativity is naturally satisfied and there is no paradox; *the 4D torque N is the same 4D quantity in all relatively moving inertial frames of reference*. Note that N will be the same 4D quantity, as in (73), for all bases, e.g., the $\{r_\mu\}$ basis, and not only for the standard basis.

In the same way as for F , Eqs. (21) and (22), or for \mathcal{M} , Eqs. (25) and (26), the bivector N can be decomposed into the “space-space” torque N_s and the “time-space” torque N_t , which *together* contain the same physical information as the bivector N , and the velocity vector v of a family of observers who measures

N . In the geometric approach, i.e., in the ISR, *both N_s and N_t are equally physical 4D torques which taken together are equivalent to the 4D torque, the bivector N ,*

$$\begin{aligned} N &= (v/c) \wedge N_t + (v/c) \cdot (N_s I), \\ N_t &= (v/c) \cdot N, \quad N_s = I(N \wedge v/c), \end{aligned} \quad (74)$$

with the condition

$$N_s \cdot v = N_t \cdot v = 0; \quad (75)$$

only three components of N_s and three components of N_t are independent since N is antisymmetric. *Both, N_s and N_t depend not only N but on v as well.* The primary physical quantity with definite physical reality for the torques is the 4D torque N , whereas N_s and N_t are derived from N and v . The equations. (74) and (75) are Eqs. (13) and (14) respectively in the first paper in [14], or Eq. (2) in [13]. If N_s and N_t are written as CBGQs in the $\{\gamma_\mu\}$ basis they become

$$N_s = (1/2c)\varepsilon^{\alpha\beta\mu\nu}N_{\alpha\beta}v_\mu\gamma_\nu, \quad N_t = (1/c)N^{\mu\nu}v_\mu\gamma_\nu, \quad (76)$$

what is Eq. (3) in [13]. As seen from (76), in the frame of “fiducial” observers and in the $\{\gamma_\mu\}$ basis, $v^\mu = (c, 0, 0, 0)$, $N_s^0 = N_t^0 = 0$ and *only the spatial components N_s^i and N_t^i remain*

$$N_s^0 = 0, \quad N_s^i = (1/2)\varepsilon^{0ijk}N_{jk}, \quad N_t^0 = 0, \quad N_t^i = N^{0i}, \quad (77)$$

which explains the names the “space-space” torque for N_s and the “time-space” torque for N_t . The quotation marks stand because such an identification of the components of the torques N_s and N_t with the components of the bivector N is not possible for some other bases. In the ISR, *both vectors N_s and N_t have to be treated on an equal footing.* It is worth noting that the whole discussion with the torque can be completely repeated for the angular momentum replacing N , N_s and N_t by J , J_s and J_t , see Eqs. (17) - (19) in the first paper in [14]. The Trouton-Noble paradox, Jackson’s paradox and Mansuripur’s paradox [34], all of them stem from the fact that in the usual approaches an independent physical reality is attributed only to N_s and J_s , or better to say, to the 3D torque \mathbf{T} and the 3D angular momentum \mathbf{L} , but not to N_t and J_t . Furthermore, in the usual approaches, the AT of \mathbf{T} and \mathbf{L} are considered to be the relativistically correct LT, see, e.g., Jackson’s paper [51] and the discussion in Sec. 3 in the first paper in [14].

Let us determine N_s and N_t for our case as 4D CBGQs in the rest frame S' . In S' , one finds from (71) and (76) that

$$N_s = N_s'^\mu \gamma'_\mu = (1/c)E'^1 p'^3 v'^0 \gamma'_2, \quad N_t = N_t'^\mu \gamma'_\mu = -(1/c^2)E'^1 m'^2 v'^0 \gamma'_3, \quad (78)$$

where, in S' , $v'^\mu = (c, 0, 0, 0)$. The “time-space” torque N_t in (78), which comes from the first term in (71), corresponds to the expression $\mathbf{R} = \mathbf{m} \times \mathbf{E} = -mE\hat{\mathbf{y}}$ in Cross [36] that describes the interaction of the magnetic moment with the electric field in the rest frame S' . (Remember that in Cross [36] the rest frame

is with unprimed quantities and the motion is along the x^3 axis.) The “space-space” torque N_s in (78), which comes from the second term in (71), does not appear in any previous paper since it emerges from the existence of the EDM p for a stationary permanent magnet, which is first predicted in Sec. 8 here. It describes the interaction of the EDM $p = p'^3 \gamma'_3$ of the stationary permanent magnet with the electric field $E = E'^1 \gamma'_1$ in the rest frame S' . In the usual formulation with the 3-vectors it would correspond to the usual 3D torque $\mathbf{T} = \mathbf{p} \times \mathbf{E}$, but, in contrast to all previous formulations, this torque is in the rest frame S' .

Next, we determine N_s and N_t in S , the lab frame. One way is to start with Eq. (78) and then to transform by the LT *all* quantities which determine N_s and N_t in (78), i.e., E'^μ , m'^μ , p'^μ , v'^μ and γ'_μ , from S' to S . This yields

$$N_s = N_s^\mu \gamma_\mu = (1/\gamma) E^1 p^3 \gamma_2, \quad N_t = -(1/c\gamma) E^1 m^2 \gamma_3. \quad (79)$$

Another way to determine N_s and N_t in S is to use the expression for N in S (72) and the relations (76). Note that “fiducial” observers are moving in S . Therefore the components v^μ of their velocity in S , which are obtained by the LT from $v'^\mu = (c, 0, 0, 0)$, are $v^\mu = (\gamma c, \gamma \beta c, 0, 0)$. Of course, for the whole CBGQ v it holds that $v = v'^\mu \gamma'_\mu = v^\mu \gamma_\mu$. Similarly, in this geometric approach, e.g., $N_s'^\mu \gamma'_\mu$ transforms under the LT as every vector transforms, i.e., as in LT (43), which means that *components $N_s'^\mu$ of the “space-space” torque N_s transform to the components N_s^μ of the same torque N_s in the S frame; there is no mixing with the components of the “time-space” torque N_t*

$$N_s^0 = \gamma(N_s'^0 + \beta N_s'^1), \quad N_s^1 = \gamma(N_s'^1 + \beta N_s'^0), \quad N_s^{2,3} = N_s'^{2,3} \quad (80)$$

and the same for N_t^μ . The LT (80) of the components of N_s and the same for N_t are obtained in the same way as the LT (43) are obtained, i.e., that both N and v from the definitions (74) are transformed by the LT R , Eq. (39), as in (40) and (41). This is in a sharp contrast to the AT (66) in which the transformed components T_i are expressed by the mixture of components T'_k of the 3D vector \mathbf{T}' and of components R'_k of another 3D vector \mathbf{R}' from the rest frame. The AT (66) can be obtained in the same way as the AT (46) and (47) in Sec. 6 are obtained, i.e., that only N from the definitions (74) is transformed by the LT R , Eq. (39), but not the velocity of the observer v . Furthermore, N_s and N_t are geometric quantities in the 4D spacetime since the components N_s^μ and N_t^μ are multiplied by the unit vectors γ_μ , whereas the 3D torque \mathbf{T} is a geometric quantity in the 3D space. It is formed multiplying the components $N^{\mu\nu}$ (i.e., T_i determined by the same identification as in (3)) of a 4D geometric quantity, the bivector N , by the unit 3D vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

It can be easily proved from (78) and (79) that the CBGQs $N_s'^\mu \gamma'_\mu$ and $N_s^\mu \gamma_\mu$ are *the same quantity N_s in S' and S frames*, and the same for N_t

$$N_s = N_s'^\mu \gamma'_\mu = N_s^\mu \gamma_\mu, \quad N_t = N_t'^\mu \gamma'_\mu = N_t^\mu \gamma_\mu; \quad (81)$$

remember that $E'^1 = (1/\gamma) E^1$. This again shows, as in [13] and [14], that *in the approach with the 4D torques N_s and N_t the principle of relativity is*

naturally satisfied and there is no paradox. Observe that N_s is determined in all relatively moving inertial frames of reference by the interaction of the EDM p of the permanent magnet and the electric field E , whereas N_t is determined by the interaction of m and E .

It is worth mentioning that, in contrast to the 3D torque \mathbf{T} , the 4D torque N_s is the same quantity for observers in S' and S even if they use different bases, e.g., $\{\gamma_\mu\}$, $\{r_\mu\}$

$$N_s = N_s^\mu \gamma_\mu = N_s'^\mu \gamma'_\mu = N_{s,r}^\mu r_\mu = N_{s,r}'^\mu r'_\mu, \quad (82)$$

where the primed quantities are the Lorentz transforms of the unprimed ones. The same holds for N_t . For the $\{r_\mu\}$ basis, this can be proved using the transformation matrix R^μ_ν from Sec. 3.1 or, as discussed in connection with Eq. (49), using the LT in the $\{r_\mu\}$ basis.

Let us suppose for a moment that a permanent magnet possesses only a MDM m and not an EDM p . This is as in the usual approaches, but we deal with correctly defined vectors in the 4D spacetime and with their LT and not with the 3-vectors and their AT. In that case, as can be seen from (78), (79) and (81), (82) in the rest frame S' and in the lab frame S as well, i.e., *in all relatively moving inertial frames of reference and for all bases in them, the “space-space” torque $N_s = 0$ and only remains the “time-space” torque N_t ,*

$$p = 0; \quad N_s = 0, \quad N_t \neq 0. \quad (83)$$

As already stated, in S' , the torque N_t corresponds to the expression $\mathbf{R} = \mathbf{m} \times \mathbf{E} = -mE\hat{\mathbf{y}}$ in Cross [36]. In all other papers from [34-36] there is no 3D torque in the rest frame S' . However, as a result of the AT of the 3-vectors (18), or (10), i.e., the AT of components (66) as in Cross [36], there is the usual 3D torque \mathbf{T} in S . In the considered case ($p = 0$), in the approach with N_s and N_t , according to (79) and (81) $N_s = 0$ and N_t is the same as in S' , which means that there is no paradox. In the formulation with N , the relations (71), (72) and (73) again show that for $p = 0$ the torque N is the same 4D quantity in all relatively moving inertial frames of reference and for all bases chosen in them and again there is no paradox. Also, as in the case with $p \neq 0$, there is no need either for the “hidden” mechanical angular momentum or for the “hidden” torque.

10. On the Aharonov-Bohm effect in terms of fields. Is the AB effect purely quantum mechanical in nature?

If the existence of the electric fields from a stationary permanent magnet would be experimentally proved then it would enable a new interpretation of the particle interference experiments, particularly of the Aharonov - Bohm (AB) effect [53]. Such electric fields offer a new possibility for the explanation of the experimentally observed fringe shift for the magnetic AB effect even in Tonomura’s experiments [54] *in terms of forces, which so far have been overlooked.* In this paper only a qualitative consideration will be presented.

Regarding the experiments with microscopic solenoids, e.g., [55], and also the recent experiment with macroscopic solenoid [56], they can be naturally explained by the fact that *always* there is an electric field outside stationary resistive conductors carrying constant currents, i.e., by *the existence of the electric force* acting on the particle. For the existence of such external electric fields from resistive conductors see, e.g., Sec. 4 in [42] and references therein. If the experiments would be made with superconducting solenoids then again there would be the external electric field, the second-order electric field (55). Thus, even in that case, it cannot be argued that there is no force acting on the particle and consequently that the observed phase shift is entirely due to nonzero vector potential.

Let us explain the above assertions in more detail. Aharonov and Bohm [53] theoretically predicted that there is a relative phase shift between two electron beams that pass on both sides of an infinitely long, stationary, coil with a steady current. The magnetic field does not exist outside that coil. But, using quantum mechanics, Aharonov and Bohm [53] showed that the interference fringes are displaced proportionally to the magnetic flux Φ_B flowing inside the coil even though neither electron beam touch the magnetic flux. They asserted: “We shall show that, contrary to the conclusions of classical mechanics, there exist effects of potentials on charged particles, even in the region where *all the fields* (my emphasis) (and therefore the forces on the particles) vanish.” Thus, according to them, the electron wave packets are influenced although *it is supposed that they travel through regions entirely free from electromagnetic fields*. As the vector potential \mathbf{A} exists in the considered case even in the magnetic field-free regions Aharonov and Bohm [53] proposed that “in quantum mechanics, the fundamental physical entities are the potentials.” Their results became very important in the forefront of physics. Vector potentials are generalized to gauge fields and these fields are considered to be the fundamental physical quantities in the modern theories of gauge fields.

They, [53], have found that the relative phase shift, $\Delta\varphi$, is produced between the two wave packets due to the vector potential as

$$\Delta\varphi = (e/\hbar) \oint \mathbf{A} \cdot d\mathbf{l} = (e/\hbar) \int \mathbf{B} \cdot d\mathbf{S} = (e/\hbar) \Phi_B, \quad (84)$$

where \mathbf{B} is the magnetic field of the solenoid.

Instead of an infinite coil with current the experiments, e.g., [55], dealt with microscopic solenoids. Möllenstedt and Bayh, [55], observed a fringe shift that is in agreement with the relation (84). The overlap between the incident electrons and the *magnetic* field strengths in their experiments was fairly small. Therefore, the experiments [55] are usually considered as a convincing demonstration of the existence of the AB effect, i.e., that the relative phase shift is due only to the vector potential and, according to (84), due to the quantum action of the magnetic flux enclosed between the two interfering electron trajectories. Thus, according to the presently accepted formulation the AB phase shift is caused by an enclosed magnetic flux and there is no need to examine any interaction between the passing charges and the sources of the magnetic flux.

However, several authors questioned the existence of the AB effect and tried to explain the observed fringe shift in a classical way, i.e., in terms of fields and the Lorentz force. One reason for such a possibility is the unphysical character of the infinitely long solenoid that is commonly used in the discussions of the AB effect. Hence, the experimental results with finite solenoids, e.g., in [55], and also with finite whiskers [57], could be attributed to the magnetic flux leaking outside such solenoids or whiskers.

In Tonomura's experiments [54] a tiny toroidal magnet is used instead of straight solenoids. The magnet was covered with a superconductor layer and further with a copper layer. The phase shift between two waves passing through the hole and outside of the toroid was measured by means of electron holography. The AB phase shift was detected even though the magnetic field was confined to the toroidal magnet; the Meissner effect prevented any flux from leaking out. The copper outer layer prevented any electrons from penetrating the magnet itself, i.e., there was no overlap of the incident electron wave with the magnetic fields inside the sample. Thus, it is generally accepted that Tonomura's experiments [54] give conclusive evidence for the AB effect, i.e., that there is no force acting on the particle and consequently that the observed phase shift is entirely due to nonzero vector potential. For a general review see, e.g., [58].

It is very interesting that both in the theoretical discussions and in the experiments with microscopic solenoids, and also in the recent experiment with macroscopic solenoid [56], it is never noticed that *always* there is an electric field outside stationary resistive conductors carrying constant currents. In such ohmic conductors there are quasistatic surface charges, which generate not only the electric field inside the wire driving the current, *but also a time independent electric field outside it*, see, e.g., Sec. 4 in [42] and references therein. There are no analytic solutions for these surface charges and the electric fields outside the wire for the case of finite solenoids. It is very difficult to determine the distribution of surface charges on the conductors of a circuit because it depends on the geometry of the circuit itself and of its surroundings. In order to have some qualitative orientation about that external electric field one can look at Eq. (16) in [42]. It is derived for a cylindrical wire of finite length. From that equation it is visible that *the electric field outside the wire is proportional to the current*. The radial component of that field falls as $1/r$. The proportionality of the external electric field with the current in the solenoid explains all important features of, e.g., the experiments from [55]. The magnetic flux Φ_B flowing inside the solenoid is also proportional to the current and it explains the generally accepted belief that the fringe shift in the experiments with microscopic solenoids is determined by the magnetic flux inside the solenoid. However, as seen from the above discussion, the relative phase shift *is not* due to the vector potential, i.e., according to (84), due to the quantum action of the magnetic flux, but *it is due to the existence of the electric field outside stationary solenoids with steady currents*.

From the viewpoint of the approach with 4D geometric quantities the *components* of the electric field 3-vector in Eq. (16) in [42] have to be understood as the *spatial components* in the standard basis of the electric field vector, a

4D geometric quantity; the rest frame of the considered wire is taken to be the γ_0 -frame and therefore the temporal component $E^0 = 0$ (also $B^0 = 0$). Similarly, the surface charge density from Eq. (12) in [42] has to be understood as the *temporal component* in the standard basis of the current density vector. In the rest frame of the wire the *spatial components are zero*, $j^i = 0$. Also, in our approach, the vector of the Lorentz force K is given by the expression (63).

Observe that in [42] two other contributions to the external electric field are mentioned in Secs. 3 and 5, but they are of no concern for our consideration. In Sec. 3 in [42] the electric field from the induced charges is investigated, but it is the same whether or not there is current in the wire. In Sec. 5 in [42] the electric field proportional to the square of the current is derived using an action-at-a-distance Weber's electrodynamics, which is not in agreement with the field theory, i.e., with the special relativity. On the other hand, the second-order electric field (55) is proportional to the square of the current and it is derived in a relativistically correct way. In our approach it replaces that one from Sec. 5 in [42]. The electric field (55) exists in a stationary resistive conductor carrying constant current as well, but there it is negligible in comparison with the external electric field that is caused by the distribution of surface charges on the conductor, i.e., with the field which is proportional to the current.

From the consideration in Sec. 8 it follows that, both in the Ampèrian approach and in the approach in which the intrinsic EDM vector d is determined by the new "time-space" spin vector Z , there is a time independent electric field outside stationary permanent magnet. This yields the possibility to explain the fringe shifts in Tonomura's experiments [54] by the existence of the external electric fields from a stationary permanent magnet and not, as generally accepted, by the existence of a nonzero vector potential. The title of the paper [54] is "Evidence for Aharonov-Bohm Effect with Magnetic Field Completely Shielded from Electron Wave" and it is written in the Abstract: "A toroidal ferromagnet was covered with a superconductor layer to confine the field, and further with a copper layer for complete shielding from the electron wave." Thus, in [54], the experimental setup is designed in such a way that practically the overlap between the incident electrons and the *magnetic* field strengths is negligible.

However, both layers that are used in [54] do not prevent the overlap between the wave functions of the incident electrons and *the electric field from stationary, toroidal ferromagnet*, which is predicted in Sec. 8.

A strong theoretical argument that supports the interpretation of the particle interference experiments in terms of forces and not in terms of potentials comes from the fact that, as already stated, i.e., as shown in the axiomatic formulation of the electromagnetism [12], the bivector F can be taken as the primary quantity and the field equation for F (20) is the basic equation for the whole electromagnetism; the bivector field F yields the complete description of the electromagnetic field and there is no need to introduce either the field vectors or the potentials.

Furthermore, the qualitative theoretical explanations of the quantum phase shifts in terms of classical force vectors in the Aharonov-Casher and the Röntgen effects are already given in [46, 47]. In [48], the dipole moments are quantized

according to (62) and it is shown that the expectation value for the quantum force vector is not zero in the case of the Aharonov-Casher and the Röntgen effects and in the neutron interferometry. This means that the phase shifts in these experiments are not due to force-free interaction of the dipole, i.e., they are not topological phase shifts.

11. Particle interference and Lorentz-violating electrodynamics

In the recent paper [37], under the title “Particle interference as a test of Lorentz-violating electrodynamics,” it is argued that in a Lorentz-violating model of electrodynamics [59] a magnetic solenoid generates not only a static magnetic field but also a static electric field, which acts on interfering particles producing an extra path-dependent phase. That nontopological phase is considered to be the Lorentz-violating correction to the standard topological (path-independent) Aharonov-Bohm phase.

However, as shown in the consideration in Secs. 7.1, 7.2 and 8, there is a static electric field not only outside a magnetic solenoid but also outside a stationary permanent magnet and that field is obtained in a consistent *Lorentz-covariant approach* with 4D geometric quantities. Furthermore, it is worth mentioning that in [37], and [59] as well as in almost the whole physical literature, it is believed that Maxwell’s equations with the 3-vectors are covariant under the LT, considering that the LPET (10) and (16) are the relativistically correct LT. But, the LPET (10) and (16) ARE NOT the LT and consequently, as explicitly proved in [8], Maxwell’s equations with 3-vectors ARE NOT covariant under the relativistically correct LT (42) and (43). Hence, there is no sense to develop Lorentz-violating model of electrodynamics assuming that the LPET (10) and (16) are the LT and that Maxwell’s equations with 3-vectors are covariant under the LT. It would be much more important that physics community stops to consider that in the 4D spacetime the Lorentz contraction and the dilation of time are the intrinsic relativistic effects and that the LPET (10) and (16) are the relativistically correct LT.

12. Discussion and Conclusions

The whole consideration explicitly shows that the 3D quantities $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, $\mathbf{P}(\mathbf{r},t)$ and $\mathbf{M}(\mathbf{r},t)$, \mathbf{p} and \mathbf{m} , their LPET, i.e., the AT, (10), (16), or (17), (18), and the equations with them are not well-defined in the 4D spacetime. More generally, we can conclude that *the 3D quantities do not have an independent physical reality in the 4D spacetime*. Contrary to the general belief, we find that, in the 4D spacetime, *it is not true that observers in relative motion see different fields; the transformations, Eqs. (10), (16), or (17), or, equivalently, (44) and (45), i.e., in the standard basis Eqs. (46) and (47), are not the LT but the LPET, i.e., the AT. According to the LT; Eqs. (40) - (43), the electric field E transforms only to the electric field E' and the same holds for the magnetic field B , for the vectors of the polarization P and the magnetization M and for the EDM p and the MDM m .*

As already stated, the principle of relativity is automatically included in the approach with well-defined 4D geometric quantities, i.e., in the ISR, whereas in Einstein's formulation of the special relativity [1] the principle of relativity is postulated outside the framework of a correct mathematical formulation of the theory and it is supposed that it holds for the equations, the physical laws, which are expressed in terms of the 3D quantities.

Minkowski's great discovery of the correct LT, Sec. 11.6 in [21], their generalization and the explicit forms (42) and (43) that are found in [6-11] and also the mathematical argument from [19] that space and time dependent electric and magnetic fields cannot be the usual 3-vectors strongly suggest the need for further critical examination of the usual formulation of electromagnetism with 3-vectors $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, $\mathbf{P}(\mathbf{r},t)$ and $\mathbf{M}(\mathbf{r},t)$ and their LPET (10), (16), or (17). It also suggests the possibility for a complete and relativistically correct formulation of classical and quantum electromagnetism with multivector fields (as physically real fields), which are defined on the 4D spacetime and which transform according to the correct LT (40) - (43).

The advantages of such formulation with 4D geometric quantities, i.e., of the ISR, are already revealed in the cases of the interaction between the dipole moment tensor D^{ab} and the electromagnetic field F^{ab} in [48] and in much more detail in [33], in the discussion of quantum phase shifts in [46, 47], in the discussion of shortcomings in the current EDM searches in [49] and in the formulation of Majorana form of the Dirac-like equation for the free-photon [60].

Particularly important results of the 4D geometric approach, i.e., of the ISR, that are reported in this paper, Secs. 7.1, 7.2, 8, refer to the existence of the second-order electric field outside a superconducting loop with steady current and to the new prediction of the electric field outside a stationary permanent magnet, i.e. to the prediction that a stationary permanent magnet possesses an intrinsic polarization, which induces the external electric field. Also, it is suggested that the measurements of that external electric field from a stationary permanent magnet could be performed by the same method with cold ions as in [31] and possibly as in [43].

The investigation of the "charge-magnet paradox" from Secs. 9.1 and 9.2 again shows that the relativistically correct description of physical phenomena without any paradoxes can be achieved in the consistent way with 4D geometric quantities as physical quantities in the 4D spacetime. On the other hand, the use of 3D quantities and their AT necessarily leads to different ambiguities and inconsistencies.

A qualitative explanation of the AB effect from Sec. 10, together with possibly positive experimental results for the existence of the electric field vector from stationary permanent magnet, will surely be important for better understanding of the classical limit of the quantum physics. A more quantitative calculation of the AB effect in terms of fields will be done, e.g., using the method from [61] and the decomposition of F (21).

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Appendix

In this Appendix we shall briefly describe the essential differences between the 4D geometric approach, the ISR, and Einstein's definition of the Lorentz contraction, e.g., for a moving rod. This is explained in detail in Secs. 2 - 2.3 in [27] and Secs. 3.1, 4.1 and Figs. 1 and 3 in [16]. Here, the notation is slightly different than in [27] and [16]. In the geometric approach one deals with the abstract 4D geometric quantities, i.e., with the position vectors x_A , x_B , of the events A and B , respectively, with the distance vector $l_{AB} = x_B - x_A$ and with the spacetime length, $l = L_0$, see (86). The essential feature of the geometric approach is that *any abstract 4D geometric quantity*, e.g., the distance vector $l_{AB} = x_B - x_A$, *is only one quantity, the same quantity in the 4D spacetime* for all relatively moving frames of reference and for all systems of coordinates that are chosen in them. The abstract vector l_{AB} can be decomposed in different bases and then these representations, the CBGQs, of the same abstract 4D geometric quantity l_{AB} contain both the basis components and the basis vectors. Let us explain it taking a particular choice for l_{AB} , which in the usual "3+1" picture corresponds to a rod that is at rest in an inertial frame of reference (IFR) S (with the standard basis in it) and situated along the common x^1 , x'^1 - axes. Its rest length is denoted as L_0 . The situation is depicted in Fig. 1 in [16]. l_{AB} is decomposed, i.e., it is written as a CBGQ, in the standard basis and in S and S' , where the rod is moving, as

$$l_{AB} = l_{AB}^\mu \gamma_\mu = 0\gamma_0 + L_0\gamma_1 = l_{AB}^{\prime\mu} \gamma'_\mu = -\beta\gamma L_0\gamma'_0 + \gamma L_0\gamma'_1, \quad (85)$$

As already stated several times, the components l_{AB}^μ are transformed by the LT and the basis vectors γ_μ by the inverse LT leaving the whole CBGQ unchanged. In S , the position vectors $x_{A,B}$ are determined simultaneously, $x_B^0 - x_A^0 = l_{AB}^0 = 0$, i.e., the temporal part of l_{AB}^μ is zero. In the standard basis, which is commonly used in the usual approaches, there is a dilation of the spatial part $l_{AB}^1 = \gamma L_0$ with respect to $l_{AB}^1 = L_0$ and not the Lorentz contraction as predicted in Einstein's formulation of special relativity. Similarly, as explicitly shown in [27] and [16], in the $\{r_\mu\}$ basis, i.e., with the "r" synchronization, if only spatial parts of $l_{AB,r}^\mu$ and $l_{AB,r}^{\prime\mu}$ are compared then one finds the dilation $\infty \succ l_{AB,r}^1 \geq L_0$ for all β_r . However, the comparison of only spatial parts of the components of the distance vector l_{AB} in S and S' is physically meaningless

in the geometric approach, since *some components of the tensor quantity, when they are taken alone, do not correspond to some definite 4D physical quantity.* Note that if $l_{AB}^0 = 0$ then the LT yield that l_{AB}^μ in any other IFR S' contains the time component as well, $l_{AB}^{\prime 0} = x_B^0 - x_A^0 = -\beta\gamma L_0 \neq 0$. Hence, *the LT yield that the spatial ends of the rod are not determined simultaneously in S' , i.e., the temporal part of l_{AB}^μ is not zero.* For the spacetime length l it holds that

$$l^2 = |l_{AB}^\mu l_{AB,\mu}| = |l_{AB}^{\prime\mu} l_{AB,\mu}'| = |l_{AB,r}^\mu l_{AB,r,\mu}| = L_0^2. \quad (86)$$

In S , the rest frame of the rod, where the temporal part of l_{AB}^μ is $l_{AB}^0 = 0$, the spacetime length l is a measure of the spatial distance, i.e., of the rest spatial length of the rod, as in the prerelativistic physics. *The observers in all other IFRs will “look” at the same events A and B , the same distance vector l_{AB} and the same spacetime length l , but associating with them different coordinates; it is the essence of the geometric approach. They all obtain the same value l for the spacetime length, $l = L_0$.*

It is worth mentioning, once again, that the 4D geometric treatment with l_{AB} and l is a generalization and a mathematically better founded formulation of the ideas expressed by Rohrlich [15] and Gamba [40]. Indeed, Rohrlich [15] states: “A quantity is therefore physically meaningful (in the sense that it is of the same nature to all observers) if it has tensorial properties under Lorentz transformations.” Similarly Gamba [40], when discussing the sameness of a physical quantity (for example, a *nonlocal quantity* $A_\mu(x_\lambda, X_\lambda)$, which is a function of two points in the 4D spacetime x_λ and X_λ) for different inertial frames of reference S and S' , declares: “The quantity $A_\mu(x_\lambda, X_\lambda)$ for S is the same as the quantity $A'_\mu(x'_\lambda, X'_\lambda)$ for S' when all the primed quantities are obtained from the corresponding unprimed quantities through Lorentz transformations (tensor calculus).” Rohrlich and Gamba worked with the usual covariant approach, i.e., with the components implicitly taken in the standard basis, which means that only Einstein’s synchronization is considered to be physically admissible. The quantities $A_\mu(x_\lambda, X_\lambda)$ and $A'_\mu(x'_\lambda, X'_\lambda)$ refer to the same physical quantity, but they are not mathematically equal quantities since bases are not included. In the approach with the 4D geometric quantities, i.e., in the ISR, one deals with mathematically equal quantities, e.g., for a nonlocal quantity $l_{AB} = x_B - x_A$ it holds that

$$l_{AB} = l_{AB}^\mu \gamma_\mu = l_{AB}^{\prime\mu} \gamma'_\mu = l_{AB,r}^\mu r_\mu = l_{AB,r}^{\prime\mu} r'_\mu = \dots, \quad (87)$$

where the primed quantities are the Lorentz transforms of the unprimed ones. In order to treat different systems of coordinates on an equal footing we have derived a form of the LT that is independent of the chosen system of coordinates, including different synchronizations, see Eq. (2) in [27], or Eq. (1) in [16]. Also, Eq. (4) in [16], it is presented the transformation matrix that connects Einstein’s system of coordinates with another system of coordinates in the same reference frame.

On the other hand, as shown in Sec. 2.2 in [27] and Sec. 4.1 and Fig. 3 in [16], in Einstein’s formulation of SR, instead of to work with geometric quantities

$x_{A,B}$, l_{AB} and l one deals only with the spatial, or temporal, *components* of their coordinate representations x_A^μ , x_B^μ and l_{AB}^μ *in the standard basis*. The geometric character of physical quantities, i.e., the basis vectors, and some asymmetric synchronization, e.g., the “r” synchronization, which is equally physical as the Einstein synchronization, are never taken into account. According to Einstein’s definition [1] of the spatial length the spatial ends of the rod must be taken simultaneously for the observer, i.e., *he defines length as the spatial distance between two spatial points on the (moving) object measured by simultaneity in the rest frame of the observer*. In the 4D (here, for simplicity, as in [27] and [16], we deal only with 2D) spacetime and in the $\{\gamma_\mu\}$ basis the simultaneous events A and B (whose spatial parts correspond to the spatial ends of the rod) are the intersections of x^1 axis (that is along the spatial basis vector γ_1) and the world lines of the spatial ends of the rod that is at rest in S and situated along the x^1 axis. The components of the distance vector are $l_{AB}^\mu = x_B^\mu - x_A^\mu = (0, L_0)$; for simplicity, it is taken that $t_B = t_A = a = 0$. Then in S , the rest frame of the object, the spatial part $l_{AB}^1 = L_0$ of l_{AB}^μ is considered to define the rest spatial length. Furthermore, one uses the inverse LT to express x_A^μ , x_B^μ and l_{AB}^μ in S in terms of the corresponding quantities in S' , in which the rod is moving. This procedure yields

$$\begin{aligned} l_{AB}^0 &= ct_B - ct_A = \gamma(l_{AB}^{\prime 0} + \beta l_{AB}^{\prime 1}), \\ l_{AB}^1 &= x_B^1 - x_A^1 = \gamma(l_{AB}^{\prime 1} + \beta l_{AB}^{\prime 0}). \end{aligned} \quad (88)$$

Now, instead of to work with 4D tensor quantities and their LT, as in the 4D geometric approach, in the usual formulation one forgets about the transformation of the temporal part l_{AB}^0 , the first equation in (88), and considers only the transformation of the spatial part l_{AB}^1 , the second equation in (88). Furthermore, *in that relation for l_{AB}^1 one assumes that $t'_B = t'_A = t' = b$, i.e., that x_B^1 and x_A^1 are simultaneously determined at some arbitrary $t' = b$ in S'* . However, in 4D (at us 2D) spacetime such an assumption means that *in S' one does not consider the same events A and B as in S but some other two events C and D , which means that $t'_B = t'_A$ has to be replaced with $t'_D = t'_C = b$* . The events C and D are the intersections of the line (the hypersurface $t' = b$ with arbitrary b) parallel to the spatial axis x'^1 (which is along the spatial base vector γ'_1) and of the above mentioned world lines of the spatial end points of the rod. Then, in the above transformation for l_{AB}^1 (88) one has to write $x_D^1 - x_C^1 = l_{CD}^1$ instead of $x_B^1 - x_A^1 = l_{AB}^1$. The spatial parts l_{AB}^1 and l_{CD}^1 are the *spatial distances* between the events A , B and C , D , respectively. *In Einstein’s formulation, the spatial distance $l_{AB}^1 = x_B^1 - x_A^1 = L_0$ defines the spatial length of the rod at rest in S , while $l_{CD}^1 = x_D^1 - x_C^1$ is considered to define the spatial length of the moving rod in S'* . Hence, from the equation for l_{AB}^1 (88) one finds the relation between $l'^1 = l_{CD}^1$ and $l^1 = l_{AB}^1 = L_0$ as the famous formula for the Lorentz contraction of the moving rod

$$l'^1 = x_D^1 - x_C^1 = L_0/\gamma = (x_B^1 - x_A^1)/\gamma, \text{ with } t'_C = t'_D, \text{ and } t_B = t_A, \quad (89)$$

where $\gamma = (1 - \beta^2)^{-1/2}$, $\beta = U/c$ and $U = |\mathbf{U}|$; \mathbf{U} is the 3-velocity of S' relative to S . As can be nicely seen from Fig. 3 in [16], the spatial lengths L_0 and

l_{CD}^1 refer not to the same 4D tensor quantity, as in the 4D geometric approach, see Fig. 1 in [16], but to two different quantities in the 4D spacetime. These quantities are obtained by the same measurements in S and S' ; the spatial ends of the rod are measured simultaneously at some $t = a$ in S and also at some $t' = b$ in S' , and a in S and b in S' are not related by the LT or any other coordinate transformation. This means that the Lorentz contraction, as already shown by Rohrlich [15] and Gamba [40], is a typical example of an AT. *It has nothing in common with the LT of the 4D geometric quantities.* We see that in Einstein's approach [1] the spatial and temporal parts of events are treated separately, and moreover the time component is not transformed in the transformation that is called - the Lorentz contraction. Thus, contrary to the generally accepted opinion, *the Lorentz contraction is not a well-defined relativistic effect in the 4D spacetime.*

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